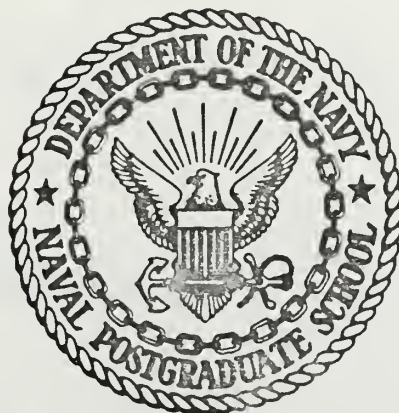# A THEORY OF GENERALIZED FILTERS

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THESIS

A THEORY OF GENERALIZED FILTERS

by

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ABSTRACT

This manuscript is a collection of many of the known results in the theory of generalized filters ($\mathfrak{Z}$ -filters) as well as an extension of some of the work in this area. The relation of $\mathfrak{Z}$ -filters to compactifications and real-compactifications is given special attention. Special emphasis is also given to the concept of tracing.

After the necessary preliminaries are disposed of, Section I motivates the study of $\mathfrak{Z}$ -filters by discussing the collection of zero-sets and then the Wallman compactification. The concept of realcompactification is also mentioned. Section II introduces the concept of $\mathfrak{Z}$ -filters and exhibits many of the elementary facts about these generalized filters.

The beginning of Section III deals with the convergence of $\mathfrak{Z}$ -filters. The Frink or Wallman-type compactification of a Tychonoff space is presented in detail. The generalization of this method in constructing realcompactifications in Tychonoff spaces is also discussed. The section closes with the presentation of some tracing results. Finally, Section IV investigates the concept of $\mathfrak{Z}$ -continuity.

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I.   INTRODUCTION

This thesis assumes the reader is familiar with the
basics of general topology. Many of the results in this
manuscript are contained in or are generalizations from A
Treatise on Realcompactness by M. Weir [33] and the notes of
R. Alo and H. Shapiro [2]. The text Rings of Continuous
Functions by L. Gillman and M. Jerison [11] provides much of
the motivation for this investigation and is referred to
frequently. When a precise reference to one of the above
works is helpful, the notation ([33], 1.3) is used to specify
Section 3 of Chapter 1 of [33].

A topological space is an ordered pair (X, τ) where X is
a non-empty set and τ is the family of all open subsets of X.
Usually the τ is suppressed and (X, τ) is denoted simply by X.
When A is a subset of a topological space X, it will be as-
sumed that A is a topological space equipped with the
relative topology $\tau_A = \{ G \cap A : G \ \varepsilon \ \tau \}$.

The power set of any set X is denoted by $\mathbb{P}(X)$. The
empty set is represented by ϕ. For a subset A of X and a
subcollection ζ of $\mathbb{P}(X)$ the trace of ζ on A is the collection
$\{ C \cap A : C \ \varepsilon \ \zeta \}$ of subsets of A and is represented by $\zeta \cap A$.
The natural numbers are denoted by N.

The interior of a subset A of a topological space X is
represented by $\text{int}_X A$, or simply int A when no confusion
results. Similarly, the closure of A is abbreviated by

$cl_X A$ or $cl\ A$. The complement of A with respect to X is
denoted by $X \setminus A$.

The topological space X is called a T_1- space in case for
each $x \varepsilon X$ the singleton set $\{x\}$ is closed. The space X is
Hausdorff when for every pair of distinct points $x,y \varepsilon X$ there
are disjoint open sets U and V such that $x \varepsilon U$ and $y \varepsilon V$. The
space X is completely regular in case for each $x \varepsilon X$ and each
closed set F which does not contain x there is a continuous
function f from X into the reals such that $f(x) = 0$ and
$f\ (F) \subset \{1\}$. Furthermore X is a Tychonoff space when X is
a completely regular T_1- space. Finally X is a normal space
in case for each pair of disjoint closed sets F_1 and F_2 there
exist disjoint open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

In a topological space X a collection $\mathcal{B}$ of closed subsets
is called a base for the closed sets if each closed set can
be written as the intersection of elements of $\mathcal{B}$. Equiva-
lently $\mathcal{B}$ is a base for the closed sets in case for each
closed set $F \subset X$ and each point $x \varepsilon X \setminus F$ there is a basic closed
set $B \varepsilon \mathcal{B}$ satisfying $F \subset B$ and $x \notin B$.

The collection of all continuous, real-valued functions
on a topological space X is denoted by $C(X)$. The set $C(X)$
can be made into a ring by defining two operations by the
formulas: $(f+g)(x) = f(x) + g(x)$ and $(fg)(x) = f(x)g(x)$
where $f,g \varepsilon\ C(X)$ and $x \varepsilon X$.

A topological space X is said to be compact in case every
open covering of X has a finite subcovering. Equivalently,

X is compact if and only if every family of closed sets with the finite intersection property has non-empty intersection ([15], 5.1).

Gilman and Jerison in ([11], 4) investigate the relationships between a Tychonoff space X that is compact and the associated algebraic ring C(X). They show the known result that <u>two compact Hausdorff spaces are homeomorphic if and only if their respective associated algebraic rings of continuous, real-valued functions are algebraically isomorphic</u>. In other words, the topology of a compact Hausdorff space X is determined by the ring C(X). This result is obtained in the following way.

For a compact space X every maximal ideal in the ring C(X) is of the form $M_X = \{ f \epsilon C(X) : f(x) = 0 \}$ for $x \epsilon X$, and these maximal ideals are distinct for distinct points of X. The collection $\mathcal{M}$ of all maximal ideals in C(X) is made into a topological space by taking as a base for the closed sets in $\mathcal{M}$ all sets of the form $\{ M_X : f \epsilon M_X \}$ for $f \epsilon C(X)$. It can be shown that $\mathcal{M}$ is a well-defined Hausdorff space and that the function g from X into $\mathcal{M}$ defined by $g(x) = M_X$ for each $x \epsilon X$ is a bijection. Notice that for any $f \epsilon C(X)$, the maximal ideal M_X belongs to the collection $B = \{ M \epsilon \mathcal{M} : f \epsilon M \}$ if and only if $f(x) = 0$. Hence, $g^{-1}[B] = \{ x \epsilon X : g(x) \epsilon B \} = \{ x \epsilon X : f(x) = 0 \}$. This observation is used to establish that the mapping g is a homeomorphism of X onto $\mathcal{M}$ ([11], 4.9).

It turns out that sets of the form $\{ x \epsilon X : f(x) = 0 \}$ for $f \epsilon C(X)$, as introduced in the preceding discussion, play

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an important role in the study of Tychonoff spaces and deserve further investigation.

Let X be a topological space and let $f \varepsilon C(X)$. The set $Z(f) = \{ x \varepsilon X : f(x) = 0 \}$ is called the zero-set of f. If $A \subset X$, then A is a zero-set in case $A = Z(f)$ for some $f \varepsilon C(X)$. The collection $\{ Z(f) : f \varepsilon C(X) \}$ of all zero-sets on X is denoted by $\mathcal{Z}(X)$. It is clear that each zero-set is closed.

Notice that $Z(f) = Z(|f|) = Z(f^n)$. If $f \equiv 0$ then $Z(f) = X$ and if $f \equiv 1$ then $Z(f) = \emptyset$. The following lemma [11] furnishes some more interesting properties of the zero-sets and provides motivation for some of the generalizations which occur later.

1.1 Lemma. Let X be a topological space and $\mathcal{Z}(X)$ be the collection of zero-sets on X. The following statements are true:

(1) The collection $\mathcal{Z}(X)$ is closed under finite unions.

(2) The collection $\mathcal{Z}(X)$ is closed under countable intersections.

(3) Every zero-set may be represented as a countable intersection of open sets.

(4) The space X is completely regular if and only if $\mathcal{Z}(X)$ is a base for the closed sets.

(5) If X is completely regular, then every neighborhood of a point contains a zero-set neighborhood of the point.

Proof: (1) Let $f, g \in C(X)$. Since $C(X)$ is a ring, $fg \in C(X)$. Moreover, it is easy to verify that $Z(f) \cup Z(g) = Z(fg)$. Hence $\mathcal{Z}(X)$ is closed under finite unions.

(2) Let $\{ f_n \in C(X) : n \in \mathbb{N} \}$ be a countable collection of elements of $C(X)$. For each $n \in \mathbb{N}$ let h_n be the constant function $h_n \equiv 2^{-n}$ and define a function g_n by $g_n = \min\{f_n, h_n\}$. From analysis $g_n \in C(X)$. Furthermore, $g(x) = \sum_{n \in \mathbb{N}} g_n(x)$ is also in $C(X)$ because $|g_n(x)| \leq 2^{-n}$ so that the above series converges uniformly. It is easy to see that

$\{ Z(f_n) : n \in \mathbb{N} \} = \{ Z(g_n) : n \in \mathbb{N} \} = Z(g)$. Hence $\mathcal{Z}(X)$ is closed under countable intersections.

(3) Let $f \in C(X)$ and let $G_n = \{ x \in X : |f(x)| < 1/n \}$ for each $n \in \mathbb{N}$. Note that each G_n is an open set. One can verify that $Z(f) = \cap \{ G_n : n \in \mathbb{N} \}$. Therefore each zero-set may be represented as a countable intersection of open sets; i.e., is a G_δ-set in X.

(4) Let X be completely regular and let F be a closed set and $x \in X \setminus F$. Then there exists a $f \in C(X)$ such that $f(x) = 1$ and $f(F) \subseteq \{0\}$. Hence $F \subseteq Z(f)$ and $x \notin Z(f)$ so that $\mathcal{Z}(X)$ is a base for the closed sets. Conversely, let $\mathcal{Z}(X)$ be a base for the closed sets and let F be a closed set with $x \in X \setminus F$. Then there is a $g \in C(X)$ such that $F \subseteq Z(g)$ and $x \notin Z(g)$. So $g(x) = c$ where c is a non-zero constant. Define the function $f = \frac{1}{c}g$. Then $f \in C(X)$ and it is easily verified that $f(x) = 1$ and $f(F) \subseteq \{0\}$. Hence X is completely regular.

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(5)   Let $x \varepsilon X$ and let $N(x)$ be a neighborhood of x.
Then there is a closed set F such that $X \setminus N(x) \subset F$ and $x \notin F$.
Since X is completely regular, there is a $f \varepsilon C(X)$ such that
$f(x) = 1$ and $f(F) \subset \{0\}$.   Let $h, k \varepsilon C(X)$ be the functions
$h \equiv 2/3$ and $k \equiv 1/3$.   Since

$$A = \{ x \varepsilon X : f(x) \geq 2/3 \} = Z(\min \{f-h, 0\}) \text{ and}$$

$$B = \{ x \varepsilon X : f(x) \leq 1/3 \} = Z(\max \{f-k, 0\}), \text{ the sets}$$

A and B are disjoint zero-set neighborhoods of $\{ x \}$ and F
respectively.   Furthermore, $A \subset N(x)$ because $X \setminus N(x) \subset B$ and
$B \cap A = \emptyset$.   Thus, $x \varepsilon$ int $A \subset A \subset N(x)$.

Prior to the introduction of zero-sets, an interesting
property of compact spaces was discussed. The compact spaces
enjoy other important properties. For instance, continuous
real-valued functions assume their infimums and supremums.
It is therefore often desirable to embed a space into a
compact space. A compactification of a space X is an ordered
pair (f, Y) where Y is a compact space and f is a homeomorphism
from X onto a dense subspace of Y. A compactification is
called Hausdorff in case Y is a Hausdorff space.

Wallman [32] developed a technique for constructing a
Hausdorff compactification of any normal space. An outline
of the essential steps in his procedure ([15], 5.R) provides
insight into later developments of this paper.

Let X be a T_1- space, let $\aleph$ be the family of all closed
subsets of X and let $\omega(\aleph)$ be the collection of all subfamilies

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τ of $\mathfrak{R}$ which have the finite intersection property and are maximal in $\mathfrak{R}$ relative to this property. If $\tau \epsilon \omega(\mathfrak{R})$, then τ is closed under finite intersections. For each closed subset F of X let $F^* = \{\tau \epsilon \omega(\mathfrak{R}) : F \epsilon \tau\}$. Then $\omega(\mathfrak{R})$ can be made into a compact topological space by taking as a base for the closed sets the collection $\{ F^* : F \epsilon \mathfrak{R}\}$. For each $x\epsilon X$, let $f(x) = \{F \epsilon \mathfrak{R} : x\epsilon F \}$. It can be shown that f is a homeomorphism of X onto a dense subspace of $\omega(\mathfrak{R})$. Moreover, if X is normal then $\omega(\mathfrak{R})$ is Hausdorff.

Frink [10] generalized Wallman's method by constructing Hausdorff compactifications for arbitrary Tychonoff spaces. He used the notion of a normal base $\mathfrak{R}$ (which will be defined later) to construct the space $\omega(\mathfrak{R})$ of all subfamilies τ of $\mathfrak{R}$ which have the finite intersection property and are maximal in $\mathfrak{R}$ relative to this property. Frink's construction will be presented in detail in Section III and motivates many of the concepts to be presented in this manuscript.

The concept of a realcompact space (originally known as a Q-space) is due to E. Hewitt [14]. The theory of realcompact spaces [33] is in many ways analogous to the theory of compact spaces. In fact, the realcompact spaces are determined by the associated algebraic ring C(X) just as the compact spaces are determined by the subring C*(X) of all bounded functions in C(X). (Note that for a compact space X the rings C(X) and C*(X) coincide since every continuous, real-valued function on a compact space is bounded.) There are many equivalent

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formulations for the concept of a realcompact space (See

[33], 2.5.1, 2.5.4, 2.5.5, 2.5.9, 2.6.4, 2.6.11, 3.11.6,

4.13.25). This manuscript will define a realcompact space

as a Tychonoff space X that can be homeomorphically embedded

as a closed subspace in a product of real lines. An equiv-

alent characterization of realcompactness will be given later

on in this work.

In [3] Alo and Shapiro use a variation of Frink's notion

of a normal base in order to construct realcompactifications

of Tychonoff spaces that are of the Wallman or Frink type.

Their considerations will also motivate many of the definitions

and results to be presented in this manuscript.

II. THE CONCEPT OF $\Re$-FILTERS

The theory of filters plays an important role in topology. A generalization of the filter concept provides a useful tool for the study of compactness and realcompactness as was mentioned earlier. This thesis assembles and extends some of the work that has been done with generalized filters and relates this work to compactifications and realcompactifications. The following definition is fundamental.

2.1 Definition. If X is a non-empty set and if $\Re \subset \mathbb{P}(X)$ is closed under finite intersections, then a collection $\mathcal{C}$ of elements in $\Re$ is a $\Re$-filter on X when the following conditions are satisfied:

   (1) The collection $\mathcal{C}$ is non-empty and $\phi \varepsilon \mathcal{C}$.

   (2) If A and B belong to $\mathcal{C}$, then $A \cap B \varepsilon \mathcal{C}$.

   (3) If $A \varepsilon \Re$ and A contains an element of $\mathcal{C}$, then $A \varepsilon \mathcal{C}$.

   If $\Re = \mathbb{P}(X)$, then $\Re$-filters are simply the familiar Bourkaki filters. Since the zero-sets are closed under countable intersections, the zero-sets may be used for $\Re$ and the corresponding zero-set-filters (or $\mathcal{Z}$-filters) are then considered. Throughout the remainder of this paper, it will be assumed that the collection $\Re$ is closed under finite intersections.

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2.2 <u>Definition</u>. A collection $\mathscr{B}$ of members of a $\mathfrak{z}$-filter $\mathfrak{T}$ is called a <u>base</u> <u>for</u> $\mathfrak{T}$ if for each $A \varepsilon \mathfrak{T}$ there is an element of $\mathscr{B}$ contained in A.

2.3 <u>Lemma</u>. <u>Let $\mathscr{B}$ be a subcollection of $\mathfrak{z}$. The collection $\mathscr{B}$ is a base for a $\mathfrak{z}$-filter $\mathfrak{T}$ if and only if both of the following hold</u>:

    (1) <u>The collection $\mathscr{B}$ is non-empty and $\phi \notin \mathscr{B}$</u> .

    (2) <u>The intersection of any two elements of $\mathscr{B}$ contains a member of $\mathscr{B}$</u> .

    <u>Proof</u>: Let $\mathscr{B}$ be a base for a $\mathfrak{z}$-filter $\mathfrak{T}$. Then $\mathscr{B}$ is non-empty by definition of a $\mathfrak{z}$-filter base and the fact that $\mathfrak{T}$ is non-empty. Moreover, $\mathscr{B} \subset \mathfrak{T}$ implies $\phi \varepsilon \mathscr{B}$. Also, from $\mathscr{B} \subset \mathfrak{T}$ and the definition of $\mathfrak{z}$-filter, $B_1, B_2 \varepsilon \mathscr{B}$ implies $B_1 \cap B_2 \varepsilon \mathfrak{T}$. Then by definition there is a member of $\mathscr{B}$ contained in $B_1 \cap B_2$.

    Conversely, suppose (1) and (2) hold. Consider $\mathfrak{T} = \{ A \varepsilon \mathfrak{z} : B \subset A, B \varepsilon \mathscr{B} \}$. If $\mathfrak{T}$ is a $\mathfrak{z}$-filter, then clearly $\mathscr{B}$ is a base for $\mathfrak{T}$. From (1) and the fact that $\mathscr{B} \subset \mathfrak{T}$, the collection $\mathfrak{T}$ is non-empty and $\phi \notin \mathfrak{T}$. The intersection of any two elements of $\mathfrak{T}$ contains the intersection of two elements of $\mathscr{B}$ which in turn contains a member of $\mathscr{B}$ by (2). It follows that the intersection of any two elements of $\mathfrak{T}$ must belong to $\mathfrak{T}$. Finally, let $Z \varepsilon \mathfrak{z}$ such that $Z \supset A$ where $A \varepsilon \mathfrak{T}$. Then there is a $B \varepsilon \mathscr{B}$ such that $Z \supset A \supset B$. Hence $Z \varepsilon \mathfrak{T}$.

Therefore, $\mathcal{T}$ is a $\mathcal{R}$-filter with base $\mathcal{B}$.

    2.3.1 <u>Remark</u>.

        If $\mathcal{B}$ is a base for a $\mathcal{R}$-filter, then the $\mathcal{R}$-filter $\mathcal{T} = \{A\varepsilon\mathcal{R} : B \subset A$ for some $B\varepsilon\mathcal{B}\}$ of the preceding proof is called <u>the</u> $\mathcal{R}$-<u>filter</u> <u>generated</u> <u>by</u> $\mathcal{B}$. Moreover, if $\mathcal{B}$ is a base for the $\mathcal{R}$-filter $\mathcal{T}$, then $\mathcal{T}$ is the $\mathcal{R}$-filter generated by $\mathcal{B}$.

        It is easy to show from the definitions that if $\mathcal{D}$ is a subcollection of $\mathcal{R}$, then there is a $\mathcal{R}$-filter containing $\mathcal{D}$ if and only if every finite subcollection of $\mathcal{D}$ has non-empty intersection.

        The following lemma will be used frequently.

    2.4 <u>Lemma</u>. <u>Let</u> $\mathcal{T}$ <u>be</u> <u>a</u> $\mathcal{R}$-<u>filter</u> <u>on</u> X <u>and</u> <u>let</u> $A\varepsilon\mathcal{R}$. <u>Then</u> $\mathcal{T} \cup \{A\}$ <u>is</u> <u>contained</u> <u>in</u> <u>some</u> $\mathcal{R}$-<u>filter</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> A <u>meets</u> <u>every</u> <u>member</u> <u>of</u> $\mathcal{T}$.

    <u>Proof</u>: The necessity is clear because the empty set cannot belong to any $\mathcal{R}$-filter. On the other hand suppose A meets every member of $\mathcal{T}$ and let $\mathcal{B} = \mathcal{T} \cap A$. By hypothesis $\emptyset \notin \mathcal{B}$, and since $\mathcal{T}$ is non-empty, the same holds true of $\mathcal{B}$. The intersection of two elements of $\mathcal{B}$ belongs to $\mathcal{B}$ because $\mathcal{T}$ is closed under finite intersections. Hence by (2.3), $\mathcal{B}$ generates a $\mathcal{R}$-filter $\mathcal{D}$ satisfying $\mathcal{T} \subset \mathcal{D}$ and $A\varepsilon\mathcal{D}$.

        An examination of the trace of $\mathcal{R}$-filters reveals difficulties not present in the study of Bourbaki filters. One

question which arises is what should be taken as the distin-
guished collection of subsets of A. Suppose for the moment
that the collection $\mathfrak{R} \cap A$ is selected. After all, the power
set of A is given by $\mathbb{P}(X) \cap A$. Recall that for a Bourbaki
filter $\mathfrak{F}$ and a subset A of X the trace of $\mathfrak{F}$ on A is a filter
if and only if A meets every member of $\mathfrak{F}$. The following
example illustrates that this result need not hold for
$\mathfrak{R}$ -filters.

Let X be the natural numbers, let the collection $\mathfrak{R}$ be
given by { {1}, {1,4}, {2,3}, {1,2,3}, $\emptyset$} and let A = {1,4}.
Notice that $\mathfrak{R}$ is closed under finite intersections and that
$\mathfrak{R} \cap A = \{ \{1\}, \{1,4\}, \emptyset \}$. The collection $\mathfrak{F} = \{ \{1,2,3\} \}$
is a $\mathfrak{R}$-filter but $\mathfrak{F} \cap A = \{\{1\}\}$ is clearly not a $\mathfrak{R} \cap A$ -
filter.

Next, suppose the collection of zero-sets is the distin-
guished collection of interest. Let X be the real line and
let A be defined as $X \setminus \{a\}$ where a is a point in X. Notice
that the set { xϵX : x>a } is a zero-set of A determined by
the continuous function f on A which takes x>a into 0 and x<a
into 1. However, f does not have a continuous extension to
all of X. In other words, there exists a zero-set on A
determined by a continuous function f which is not obtained
by the intersection of A with the zero-set of a function on the
whole space.

This example suggests the blanket assumption that $\mathfrak{R} \cap A$
be taken as the distinguished collection of subsets of A

should not be made. Throughout the remainder of this paper,
$\mathfrak{R}_A$ will denote simply an arbitrary subcollection of $\mathbb{P}(A)$.
The following lemmas investigate the problem of tracing with
$\mathfrak{R}$-filters. Additional results will be obtained in Section
III. The following definition will be useful.

2.5 <u>Definition</u>. Let X be a set and let $\mathfrak{R} \subset \mathbb{P}(X)$. The
collection $\mathfrak{R}$ is a <u>ring of sets</u> in case it is closed under
finite unions and also finite intersections.

The power set of any set is clearly a ring of sets.
From (1.1) the collection of zero-sets for any X is also a
ring of sets.

2.6 <u>Lemma</u>. <u>Let</u> A $\subset$ X <u>and let</u> $\mathfrak{T}$ <u>be a</u> $\mathfrak{R}$-<u>filter with base</u>
$\mathfrak{B}$ <u>such that</u> $\mathfrak{R}_A \supset \mathfrak{T} \cap A$. <u>Then the following statements are</u>
<u>equivalent</u>:

       (1) <u>The set</u> A <u>meets every member of</u> $\mathfrak{T}$.

       (2) <u>The collection</u> $\mathfrak{T} \cap A$ <u>generates a</u>
           $\mathfrak{R}_A$-<u>filter</u>.

       (3) <u>The collection</u> $\mathfrak{B} \cap A$ <u>is a</u> $\mathfrak{R}_A$-<u>filter</u> <u>base</u>.

       <u>Proof</u>: (1) implies (2). The collection $\mathfrak{T} \cap A$ is
non-empty by hypothesis since $\mathfrak{T}$ is non-empty. By (1) the
empty set does not belong to $\mathfrak{T} \cap A$. Finally the intersection
of two members of $\mathfrak{T} \cap A$ belongs to $\mathfrak{T} \cap A$ since $\mathfrak{T}$ is closed
under finite intersections. Hence, by (2.3) the collection
$\mathfrak{T} \cap A$ generates a $\mathfrak{R}_A$-filter.

(2) implies (3). Since $\mathcal{B}$ is non-empty, the same is true of $\mathcal{B} \cap A$. The empty set is not in $\mathcal{B} \cap A$ because $\mathcal{B} \subset \mathcal{F}$ and $\mathcal{F} \cap A$ generates a $\mathcal{R}_A$-filter. The intersection of two elements of $\mathcal{B} \cap A$ contains a member of $\mathcal{B} \cap A$ since the intersection of any two elements of $\mathcal{B}$ contains an element of $\mathcal{B}$.

(3) implies (1). Any $F \in \mathcal{F}$ contains a $B \in \mathcal{B}$ by definition of filter base. Hence $F \cap A$ is non-empty because $F \cap A \supset B \cap A$, and $B \cap A$ is non-empty by (3).

2.7 <u>Lemma</u>. Let $A \subset X$, $\mathcal{R}$ <u>be a ring of sets</u>, $\mathcal{R}_A \subset \mathcal{R} \cap A$, <u>and let</u> $\mathcal{F}$ <u>be a</u> $\mathcal{R}$-<u>filter with base</u> $\mathcal{B}$ <u>satisfying</u> $\mathcal{F} \cap A \subset \mathcal{R}_A$. <u>If</u> $\mathcal{B} \cap A$ <u>is a</u> $\mathcal{R}_A$-<u>filter base on</u> A, <u>then</u> $\mathcal{B} \cap A$ <u>generates the</u> $\mathcal{R}_A$-<u>filter</u> $\mathcal{F} \cap A$.

<u>Proof</u>: Let $\mathcal{H}$ be the $\mathcal{R}_A$-filter generated by $\mathcal{B} \cap A$. For any $F \in \mathcal{F}$ there is a $B \in \mathcal{B}$ such that $F \supset B$. Then $F \cap A \supset B \cap A$ so that $F \cap A \in \mathcal{H}$. Hence $\mathcal{F} \cap A \subset \mathcal{H}$. On the other hand, let $G \in \mathcal{H}$. Then there is a $B \in \mathcal{B}$ such that $G \supset B \cap A$. Furthermore, since $\mathcal{R}_A \subset \mathcal{R} \cap A$, there is a $Z \in \mathcal{R}$ such that $G = Z \cap A$. Hence $Z \cap A \supset B \cap A$. Since $\mathcal{R}$ is a ring of sets, $Z \cup B \in \mathcal{R}$. Moreover, $Z \cup B$ is necessarily in $\mathcal{F}$. Hence $(Z \cup B) \cap A = (Z \cap A) \cup (B \cap A) = Z \cap A = G$ is an element of $\mathcal{F} \cap A$. That is, $G \subset \mathcal{F} \cap A$. Therefore, $\mathcal{H} = \mathcal{F} \cap A$.

The proof of the following corollary is immediate from (2.6) and (2.7).

2.8 <u>Corollary</u>. Let $A \subset X$, $\mathfrak{Z}$ <u>be a ring of sets</u>, $\mathfrak{Z}_A \subset \mathfrak{Z} \cap A$, <u>and let</u> $\mathcal{F}$ <u>be a</u> $\mathfrak{Z}$-<u>filter with base</u> $\mathcal{B}$ satisfying $\mathcal{F} \cap A \subset \mathfrak{Z}_A$. <u>The following statements are equivalent:</u>

(1) <u>The set</u> A <u>meets every member of</u> $\mathcal{F}$.

(2) <u>The collection</u> $\mathcal{B} \cap A$ <u>is a</u> $\mathfrak{Z}_A$-<u>filter base.</u>

(3) <u>The collection</u> $\mathcal{F} \cap A$ <u>is a</u> $\mathfrak{Z}_A$-<u>filter.</u>

2.9 <u>Lemma</u>. <u>Let</u> $A \subset X$ <u>and let</u> $\mathcal{G}$ <u>be a</u> $\mathfrak{Z}_A$-<u>filter such that</u> <u>for every</u> $G \in \mathcal{G}$ <u>there is a</u> $Z \in \mathfrak{Z}$ <u>such that</u> $Z \cap A = G$. <u>If for</u> <u>every</u> $Z \in \mathfrak{Z}$ <u>that contains an element of</u> $\mathcal{G}$ <u>it is true that</u> $Z \cap A$ <u>belongs to</u> $\mathcal{G}$, <u>then there is a</u> $\mathfrak{Z}$-<u>filter</u> $\mathcal{F}$ <u>such that</u> $\mathcal{F} \cap A = \mathcal{G}$.

<u>Proof</u>: The collection $\mathcal{F} = \{ Z \in \mathfrak{Z} : Z \supset G, G \in \mathcal{G}\}$ is non-empty because $\mathcal{G}$ is non-empty and for each $G \in \mathcal{G}$ there is a $Z \in \mathfrak{Z}$ with $Z \supset G$. The empty set is not in $\mathcal{F}$ since $\phi \notin \mathcal{G}$. If $Z_1, Z_2 \in \mathcal{F}$, then there are $G_1, G_2 \in \mathcal{G}$ such that $Z_1 \supset G_1$ and $Z_2 \supset G_2$. Hence $Z_1 \cap Z_2 \supset G_1 \cap G_2$ so that $Z_1 \cap Z_2 \in \mathcal{F}$. Furthermore, if an element of $\mathfrak{Z}$ contains an element of $\mathcal{F}$, it must contain a member of $\mathcal{G}$ and thus also belong to $\mathcal{F}$. Hence, $\mathcal{F}$ is a $\mathfrak{Z}$-filter. Now, if $Z \in \mathcal{F}$ then $Z \supset G$ for some $G \in \mathcal{G}$. Then by hypothesis, $Z \cap A \in \mathcal{G}$. Hence $\mathcal{F} \cap A \subset \mathcal{G}$. On the other hand, if $G \in \mathcal{G}$ then there is a $Z \in \mathfrak{Z}$ such that $Z \cap A = G$. Hence $Z \in \mathcal{F}$ by definition. There-fore $\mathcal{F} \cap A = \mathcal{G}$.

2.10 <u>Corollary</u>. <u>If</u> $A \subset X$ <u>and if</u> $\mathcal{G}$ <u>is a</u> $\mathfrak{Z} \cap A$-<u>filter, then</u> <u>there exists a</u> $\mathfrak{Z}$-<u>filter</u> $\mathcal{F}$ <u>such that</u> $\mathcal{F} \cap A = \mathcal{G}$.

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Proof: For each $G \varepsilon \mathcal{H}$ there is a $Z \varepsilon \mathfrak{Z}$ such that $Z \cap A = G$, and for each $Z \varepsilon \mathfrak{Z}$ it is true that $Z \cap A \varepsilon \mathfrak{Z} \cap A$. If $Z \varepsilon \mathfrak{Z}$ contains a member G of $\mathcal{H}$, then $Z \cap A \supset G$ implies that $Z \cap A$ is an element of $\mathcal{H}$. Now apply (2.9).

A $\mathfrak{Z}$-filter $\mathcal{U}$ is called a $\mathfrak{Z}$-<u>ultrafilter</u> provided it is a maximal $\mathfrak{Z}$-filter with respect to the partial ordering of set inclusion in the collection of all $\mathfrak{Z}$-filter on X. In other words, if $\mathcal{T}$ is a $\mathfrak{Z}$-filter containing $\mathcal{U}$ then $\mathcal{T} = \mathcal{U}$. The following result carries over from Bourbaki filters.

2.11 <u>Lemma</u>. <u>Every</u> $\mathfrak{Z}$-<u>filter</u> <u>is</u> <u>contained</u> <u>in</u> <u>a</u> $\mathfrak{Z}$-<u>ultra-filter</u>.

Proof: Let $\mathcal{T}$ be a $\mathfrak{Z}$-filter and denote the collection of $\mathfrak{Z}$-filters containing $\mathcal{T}$ by $\mathcal{F}(X)$. Then $\mathcal{F}(X)$ is partially ordered by set inclusion. If Φ is any chain in $\mathcal{F}(X)$, then the union $\cup \Phi$ of the $\mathfrak{Z}$-filters belonging to Φ also belongs to $\mathcal{F}(X)$. This assertion is established as follows. First, $\cup \Phi$ is non-empty because Φ is non-empty. Also, $\emptyset \notin \cup \Phi$ because the empty set does not belong to any $\mathfrak{Z}$-filter in Φ. Next, for any two elements Z_1 and Z_2 of $\cup \Phi$ there are filters $\mathcal{T}$ and $\mathcal{H}$ in Φ such that $Z_1 \varepsilon \mathcal{T}$ and $Z_2 \varepsilon \mathcal{H}$. Since Φ is a chain, it may be assumed without loss of generality that $\mathcal{T} \subset \mathcal{H}$. Hence $Z_1 \cap Z_2$ belongs to $\mathcal{H}$ and therefore also belongs to $\cup \Phi$. Finally, for any $Z \varepsilon \mathfrak{Z}$ which contains an element $Z_1 \varepsilon \cup \Phi$, it is the case that Z belongs to the $\mathfrak{Z}$-filter $\mathcal{T}$ satisfying $Z_1 \varepsilon \mathcal{T}$ and $\mathcal{T} \varepsilon \Phi$. Hence the set Z is in $\cup \Phi$. Thus every chain Φ in $\mathcal{F}(X)$ has an

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upper bound in $\xi(X)$. By Zorn's Lemma the collection $\xi(X)$
has a maximal element $\mathcal{U}$. Suppose a $\mathcal{R}$-filter $\mathcal{Y}$ contains $\mathcal{U}$.
Then $\mathcal{Y}$ must contain $\mathcal{F}$ and hence belong to $\xi(X)$. Since $\mathcal{U}$ is
maximal in $\xi(X)$, the $\mathcal{R}$-filter $\mathcal{Y}$ must be precisely $\mathcal{U}$.
Therefore, $\mathcal{U}$ is a $\mathcal{R}$-ultrafilter which contains $\mathcal{F}$ which
completes the proof.

2.12 <u>Lemma</u>. Let $\mathcal{R}$ <u>be a ring of sets such that</u> $A \varepsilon \mathcal{R}$, <u>let</u>
$\mathcal{R}_A \subset \mathcal{R} \cap A$, <u>and let</u> $\mathcal{F}$ <u>be a</u> $\mathcal{R}$-<u>ultrafilter satisfying</u>
$\mathcal{F} \cap A \subset \mathcal{R}_A$. <u>Then</u> $\mathcal{F} \cap A$ <u>is a</u> $\mathcal{R}_A$-<u>filter if</u> <u>and</u> <u>only if</u> $A \varepsilon \mathcal{F}$.
<u>In this case</u> $\mathcal{F} \cap A$ <u>is, in fact, a</u> $\mathcal{R}_A$-<u>ultrafilter</u>.

<u>Proof</u>: The set A meets every member of $\mathcal{F}$ because $\mathcal{F} \cap A$ is
a $\mathcal{R}_A$-filter. By (2.4) the collection $\mathcal{F} \cup \{A\}$ is contained in
a $\mathcal{R}$-filter $\mathcal{Y}$. But $\mathcal{F}$ is maximal so $\mathcal{F} = \mathcal{Y}$. Finally, $A \varepsilon \mathcal{F}$
because $A \varepsilon \mathcal{Y}$. Conversely, if $A \varepsilon \mathcal{F}$ then A meets every member
of $\mathcal{F}$ so that $\mathcal{F} \cap A$ is a $\mathcal{R}_A$-filter by (2.8). Now suppose there
is a $\mathcal{R}_A$-filter $\mathcal{Y}$ containing $\mathcal{F} \cap A$ and let $G \varepsilon \mathcal{Y}$. By hypothesis
there is a set $Z \varepsilon \mathcal{R}$ such that $Z \cap A = G$. For every $F \varepsilon \mathcal{F}$ the inter-
section $(F \cap A) \cap (Z \cap A)$ is non-empty because $F \cap A$ belongs to $\mathcal{Y}$.
It follows that $F \cap Z \neq \phi$. Once again (2.4) may be applied
so that $\mathcal{F} \cup \{Z\}$ is contained in a $\mathcal{R}$-filter $\mathcal{V}$. As before it
follows that $\mathcal{F} = \mathcal{V}$ so $Z \varepsilon \mathcal{F}$. Therefore, $Z \cap A = G$ belongs to
$\mathcal{F} \cap A$ so $\mathcal{Y} = \mathcal{F} \cap A$. It then follows that $\mathcal{F} \cap A$ is a $\mathcal{R}_A$-ultra-
filter.

Recall that a space is compact if and only if every family
of closed sets with the finite intersection property has a

non-empty intersection. If $\mathfrak{Z}$ is a collection of closed sets, for example the zero-sets, it follows that X will be compact only if each $\mathfrak{Z}$-filter $\mathfrak{T}$ is such that $\cap\{ Z\epsilon\mathfrak{Z} : Z\epsilon\mathfrak{T} \} \neq \emptyset$ because $\mathfrak{T}$ has the finite intersection property. This observation leads to the next definition together with the lemma and corollary which follow.

2.13 Definition. A $\mathfrak{Z}$-filter $\mathfrak{T}$ is said to be <u>fixed</u> provided $\cap\{ Z\epsilon\mathfrak{Z} : Z\epsilon\mathfrak{T}\} \neq \emptyset$; otherwise, $\mathfrak{T}$ is said to be <u>free</u>.

2.14 <u>Lemma</u>. <u>A Tychonoff space X is compact if and only if every zero-set-ultrafilter</u> ($\mathcal{Z}$-<u>ultrafilter</u>) <u>on X is fixed</u>.

<u>Proof</u>: Since any $\mathcal{Z}$-ultrafilter $\mathcal{U}$ is a collection of closed sets with the finite intersection property, $\mathcal{U}$ must be fixed whenever X is compact. Now, suppose that every $\mathcal{Z}$-ultrafilter is fixed. Let $\mathcal{E}$ be a family of closed sets with the finite intersection property. By (1.1) the zero-sets are a base for the closed sets so that each member of $\mathcal{E}$ may be represented as the intersection of a collection of zero-sets. Let $\mathcal{B} = \{ Z\epsilon\ \mathcal{Z}(X) : Z\epsilon\{Z_\lambda\epsilon\ \mathcal{Z}(X) : \lambda\epsilon\Lambda \}\}$ where $C = \cap\{ Z_\lambda : \lambda\epsilon\Lambda\}$ for some $C\epsilon\mathcal{E}$. Now let $\mathcal{B}'$ consist of all finite intersections of members of $\mathcal{B}$. The collection $\mathcal{B}'$ is a $\mathcal{Z}$-filter base. This assertion is true because the finite intersection property of $\mathcal{E}$ together with the fact that each member of $\mathcal{B}$ contains an element of $\mathcal{E}$ imply that $\mathcal{B}'$ is a non-empty collection such that $\emptyset\notin\mathcal{B}'$. Moreover, by the construction of $\mathcal{B}'$ the intersection of two members of $\mathcal{B}'$ is in $\mathcal{B}'$.

Then (2.3) may be applied. It follows that $\mathcal{B}'$ is contained in some Z-ultrafilter $\mathcal{U}$ which is fixed by hypothesis. Finally, $\cap\{$ C : $C \in \mathcal{B}\} \neq \phi$ because $\cap\{$ $Z \in Z(X)$: $Z \in \mathcal{U}\} \subset$ $\{$ $Z \in Z(X)$: $Z \in \mathcal{B}'$ $\}$ $\subset$ $\{$ $Z \in Z(X)$: $Z \in \mathcal{B}\}$ $=$ $\cap\{$ C : $C \in \mathcal{C}\}$ which completes the proof.

   2.15 <u>Corollary</u>. A chonoff space X <u>is</u> <u>compact</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> <u>every</u> Z-<u>filter</u> <u>is</u> <u>fixed</u>.

   <u>Proof</u>: The proof simply consists of using (2.14) together with the fact that every Z-filter is contained in a Z-ultrafilter.

   In Section I the Wallman compactification was discussed. Observe that the collection $\omega(\mathfrak{Z})$ mentioned there is actually the collection of all $\mathfrak{Z}$-ultrafilters on X where $\mathfrak{Z}$ consists of all the closed subsets of X. Moreover, the function f from X into $\omega(\mathfrak{Z})$ defined by $f(x) = \{$ $Z \in \mathfrak{Z}$: $x \in Z$ $\}$ clearly maps each point of X into a fixed $\mathfrak{Z}$-ultrafilter.

   As has been noted, Frink [10] generalized Wallman's method to obtain compactifications of arbitrary Tychonoff spaces. His approach was to impose certain properties on $\mathfrak{Z}$ $\subset \mathbb{P}(X)$ that are analogous to those enjoyed by the collection of closed sets in a normal space. It is then possible to make $\omega(\mathfrak{Z})$ into a compact Hausdorff space such that the points of X can be naturally identified with the fixed $\mathfrak{Z}$-ultrafilters. Furthermore, it can be shown that $\omega(\mathfrak{Z})$ contains a dense homeomorphic image of X.

When X is a T_1-space, each point in X is a member of the
collection of all closed sets. Moreover, if $x \varepsilon X$ and $Z \subset X$
is a closed set such that $x \notin Z$, then there is a closed set
(namely $\{x\}$) such that $\{x\} \cap Z = \phi$. These facts are used in
the Wallman compactification to show that for any $x \varepsilon X$ the
collection $\{ Z : Z$ is closed, $x \varepsilon Z \}$ is in $\omega(\mathfrak{Z})$. This
observation motivates the introduction of the following con-
cepts which enable a similar result to be obtained for a more
arbitrary $\mathfrak{Z}$.

2.16 <u>Definition</u>. For a non-empty set X, a non-empty
collection $\mathfrak{Z} \subset \mathbb{P}(X)$ is said to be $\mathfrak{Z}$-<u>disjunctive</u> if for each
set $Z \varepsilon \mathfrak{Z}$ and point $x \notin Z$ there exists a $Z_1 \varepsilon \mathfrak{Z}$ such that $x \varepsilon Z_1$ and
$Z \cap Z_1 = \phi$. If X is a topological space, then $\mathfrak{Z}$ is said to be
<u>disjunctive</u> in case for each closed set $F \subset X$ and point $x \notin F$
there exists a $Z \varepsilon \mathfrak{Z}$ such that $x \varepsilon Z$ and $Z \cap F = \phi$.

Note that if $\mathfrak{Z}$ is a disjunctive collection of closed sets
of X, then $\mathfrak{Z}$ is $\mathfrak{Z}$-disjunctive. It is clear that in a T_1-space
the collection of all closed sets is disjunctive as well as
$\mathfrak{Z}$ -disjunctive. The collection $\mathcal{Z}(X)$ of all zero-sets on a
Tychonoff space X is disjunctive and since such sets are
closed, $\mathcal{Z}(X)$ is also $\mathfrak{Z}$-disjunctive. Also, note that the
power set of X is $\mathfrak{Z}$-disjunctive, and if X is a T_1-space then
the power set is disjunctive. The importance of these results
is revealed in the following propositions.

2.17 <u>Lemma</u>. <u>If the collection</u> $\mathcal{Z}$ <u>is</u> $\mathcal{Z}$-<u>disjunctive then</u> <u>for any point</u> xϵX <u>the collection</u> $\tau_x = \{\ Z\epsilon\mathcal{Z} : x\epsilon Z\ \}$ <u>is</u> <u>a fixed</u> $\mathcal{Z}$-<u>ultrafilter</u>. <u>Throughout the remainder of this</u> <u>paper the notation</u> τ_x <u>will refer to this particular</u> $\mathcal{Z}$ -<u>ultrafilter</u>.

 <u>Proof</u>: Let $Z\epsilon\mathcal{Z}$ and suppose $Z\notin\tau_x$. Then x$\notin$Z so there exists a $Z_1\epsilon\mathcal{Z}$ such that xϵZ_1 and $Z\cap Z_1 = \phi$ by the $\mathcal{Z}$ -disjunctive property. It follows that τ_x is non-empty. The empty set is not in τ_x because the point x belongs to every element of τ_x. The intersection of any two members of τ_x contains the point x and therefore belongs to τ_x. For any $Z\epsilon\mathcal{Z}$ that contains an element of τ_x the point x must belong to Z and hence $Z\epsilon\tau_x$. Clearly, $x\epsilon\cap\{Z\epsilon\mathcal{Z}: Z\epsilon\tau_x\}$.

 Now suppose that $\mathcal{Y}$ is a $\mathcal{Z}$-filter which contains τ_x. If B is a member of $\mathcal{Y}$ and $B\notin\tau_x$, then x$\notin$B. Hence there is a $Z\epsilon\mathcal{Z}$ such that xϵZ and $Z\cap B = \phi$. Thus, Z belongs to τ_x and hence to $\mathcal{Y}$. This contradicts the finite intersection property of $\mathcal{Y}$. Hence $\mathcal{Y} = \tau_x$ which completes the proof.

 2.18 <u>Lemma</u>. <u>Let the collection</u> $\mathcal{Z}$ <u>be</u> $\mathcal{Z}$-<u>disjunctive and let</u> $\mathcal{U}$ <u>be a</u> $\mathcal{Z}$-<u>ultrafilter</u>. <u>Then</u> $\mathcal{U}$ <u>is fixed if and only if</u> $\mathcal{U} = \tau_x$ <u>for some</u> xϵX.
 <u>Proof</u>: Let $\mathcal{U}$ be fixed and let $x\epsilon\cap\{\ Z\epsilon\mathcal{Z} : Z\epsilon\mathcal{U}\}$. Since $\mathcal{Z}$ is $\mathcal{Z}$ -disjunctive, τ_x is a $\mathcal{Z}$-ultrafilter. For any $Z\epsilon\mathcal{U}$ the point xϵZ so that $Z\epsilon\tau_x$. It follows that $\mathcal{U}\subset\tau_x$. Hence $\mathcal{U} = \tau_x$ since $\mathcal{U}$ is a $\mathcal{Z}$-ultrafilter. The proof of the converse is immediate.

Whenever $\mathfrak{R}$ is $\mathfrak{R}$-disjunctive or whenever $\mathfrak{R}$ is a disjunctive collection of closed sets, the association of a point $x \varepsilon X$ with the $\mathfrak{R}$-ultrafilter $\mathfrak{T}_X$ in $\omega(\mathfrak{R})$ seems quite natural. If this association is to be used to define a mapping which embeds X into $\omega(\mathfrak{R})$, then this association must be one-to-one. These considerations motivate some of the definitions which follow in the next section.

III. CONVERGENCE and COMPACTIFICATIONS

The concept of the convergence of Bourbaki filters, or
equivalently that of the convergence of nets, plays a major
role in the study of continuity, compactness, and real-
compactness. The idea of $\mathfrak{R}$-filter convergence is utilized
in obtaining Wallman-type compactifications and realcompacti-
fications. This section examines the convergence properties
of $\mathfrak{R}$-filters and exhibits the results that are useful in
obtaining these compactifications and realcompactifications.
Recall that $\mathfrak{R}$ is always closed under finite intersections.

3.1 Definition. Let $\mathcal{B}$ be a $\mathfrak{R}$-filter base on a topo-
logical space X and let xεX. The point x is said to be a
limit point of $\mathcal{B}$ in case for every neighborhood $N(x)$ of x
there exists a B$\varepsilon\mathcal{B}$ such that $B \subset N(x)$. In this case $\mathcal{B}$ is said
to converge to x. The point x is said to be a cluster point
of $\mathcal{B}$ if for each neighborhood $N(x)$ of x, $N(x) \cap B \neq \phi$ for all
Bε $\mathcal{B}$.

The easy proofs of the following facts are omitted. If
$\mathcal{T}$ is a $\mathfrak{R}$-filter with base $\mathcal{B}$, then the point xεX is a limit
point (respectively, cluster point) of $\mathcal{T}$ if and only if x is a
limit point (respectively, cluster point) of $\mathcal{B}$. Let $\mathcal{T}$ and $\mathcal{Y}$
be $\mathfrak{R}$-filters with $\mathcal{T} \subset \mathcal{Y}$. If the point x$\varepsilon$X is a limit point
of $\mathcal{T}$, then x is a limit point of $\mathcal{Y}$. Also, if x is a cluster
point of $\mathcal{Y}$, then x is a cluster point of $\mathcal{T}$. Finally, every
fixed $\mathfrak{R}$-filter has a cluster point.

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Recall that the collection of all neighborhoods of a given point x is a Bourbaki filter. Therefore, in the theory of Bourbaki filters, there always exists a Bourbaki filter converging to the point x for any xϵX. However, this is not necessarily the case for $\mathfrak{Z}$-filters.

Consider the real numbers under the discrete topology. Let x and y be real numbers and let $\mathfrak{Z}$ consist of those sub-sets of the reals that contain both of the points x and y. The collection $\mathfrak{Z}$ is closed under finite intersections. Now let $\mathfrak{F}$ be any $\mathfrak{Z}$-filter that converges to x. The point x is itself a neighborhood of x and so there must be an F$\epsilon\mathfrak{F}$ such that F$\subset\{$ x $\}$. But the points x and y belong to each Z ϵ $\mathfrak{F}$ which is a contradiction. Hence, there does not exist a $\mathfrak{Z}$ -filter that converges to x. The following concept elim-inates this situation.

3.2 <u>Definition</u>. For a topological space X, the collection $\mathfrak{Z}$ is called a <u>local</u> <u>base</u> if for each point xϵX and each neighborhood N(x) of x there exists a Z$\epsilon\mathfrak{Z}$ such that . xϵ int Z $\subset$Z $\subset$N(x).

Observe that the power set of X is a local base under any topology on X. It can be shown that <u>if</u> X <u>is a</u> <u>Tychonoff</u> <u>space</u>, <u>the</u> <u>collection</u> <u>of</u> <u>all</u> <u>zero-sets</u> <u>is a local</u> <u>base</u> ([11], 3.2).

3.3 <u>Lemma</u>. <u>If</u> $\mathfrak{Z}$ <u>is a</u> <u>collection</u> <u>of</u> <u>closed</u> <u>subsets</u> <u>of</u> X <u>that</u> <u>is a</u> <u>local</u> <u>base</u>, <u>then</u> $\mathfrak{Z}$ <u>is</u> <u>disjunctive</u>.

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Proof: Let $Z \in \mathfrak{Z}$ and $x \notin Z$. Then $X \smallsetminus Z$ is open and $x \in X \smallsetminus Z$. By the local base property there is a $Z_1 \in \mathfrak{Z}$ such that $x \in \text{int } Z_1 \subset Z_1 \subset X \smallsetminus Z$. It then follows that $Z \cap Z_1 = \phi$ so that $\mathfrak{Z}$ is disjunctive.

3.4 Lemma. If $\mathfrak{Z}$ is a local base on a topological space X and $x \in X$, then $\mathcal{V}(x) = \{ Z \in \mathfrak{Z} : Z$ is a neighborhood of $x \}$ is a $\mathfrak{Z}$-filter converging to x.

Proof: The collection $\mathcal{V}(x)$ is non-empty because by the local base property there is a $Z \in \mathfrak{Z}$ such that $x \in \text{int } Z \subset Z \subset X$. The empty set is not in $\mathcal{V}(x)$ because x belongs to each member of $\mathcal{V}(x)$. Let Z and Z_1 belong to $\mathcal{V}(x)$. It follows that $x \in (\text{int } Z) \cap (\text{int } Z_1) \subset Z \cap Z_1$. Then $Z \cap Z_1$ is in $\mathcal{V}(x)$ since $(\text{int } Z) \cap (\text{int } Z_1)$ is open. It also follows that if $Z \in \mathfrak{Z}$ and if $Z \supset B$ where $B \in (x)$, then Z is a neighborhood of x. Hence, $\mathcal{V}(x)$ is a $\mathfrak{Z}$-filter. Finally, let $N(x)$ be any neighborhood of x. Then there is a $Z \in \mathfrak{Z}$ such that $x \in \text{int } Z \subset Z \subset N(x)$. The set Z must also be in $\mathcal{V}(x)$ so that x is a limit point of $\mathcal{V}(x)$.

The collection $\mathcal{V}(x)$ is called the $\mathfrak{Z}$-neighborhood filter associated with the point x. The next result gives additional information about the $\mathfrak{Z}$-filters which converge to a particular point and is used in later work.

3.5 Lemma. For a topological space X, let $\mathfrak{Z}$ be a local base, let $\mathcal{T}$ be a $\mathfrak{Z}$-filter, and let $x \in X$. The following statements are true:

(1) The <u>point</u> x <u>is</u> <u>a</u> <u>limit</u> <u>point</u> <u>of</u> $\mathcal{T}$ <u>if</u> <u>and</u> <u>only</u>
<u>if</u> $\mathcal{V}(x) \subset \mathcal{T}$.

(2) The <u>point</u> x <u>is</u> <u>a</u> <u>cluster</u> <u>point</u> <u>of</u> $\mathcal{T}$ <u>if</u> <u>and</u> <u>only</u>
<u>if</u> <u>there</u> <u>exists</u> <u>a</u> $\mathcal{Z}$<u>-filter</u> $\mathcal{Y}$ <u>converging</u> <u>to</u> x
<u>and</u> <u>satisfying</u> $\mathcal{T} \subset \mathcal{Y}$.

<u>Proof:</u> (1) Let x be a limit point of $\mathcal{T}$ and let $Z \varepsilon \mathcal{V}(x)$.
The set Z is a neighborhood of x so there is a set $F \varepsilon \mathcal{T}$ such
that $F \subset Z$. It follows that $Z \varepsilon \mathcal{T}$. Hence, $\mathcal{V}(x) \subset \mathcal{T}$.
Conversely, let $\mathcal{V}(x) \subset \mathcal{T}$ and let N(x) be any neighborhood of
x. By the local base property there is a $Z \varepsilon \mathcal{Z}$ satisfying
$x \varepsilon$ int $Z \subset Z \subset N(x)$. It follows that $Z \varepsilon \mathcal{V}(x)$ so that $Z \varepsilon \mathcal{T}$.
Hence, $\mathcal{T}$ converges to x.

(2) Let x be a cluster point of $\mathcal{T}$ and consider
$\mathcal{B} = \{ B \varepsilon \mathcal{Z} : B = F \cap Z, F \varepsilon \mathcal{T}, Z \varepsilon \mathcal{V}(x) \}$. The collection $\mathcal{B}$ is
non-empty since $\mathcal{T}$ and $\mathcal{V}(x)$ are both non-empty. The empty set
is not in $\mathcal{B}$ because x is a cluster point of $\mathcal{T}$ and the elements
of $\mathcal{V}(x)$ are neighborhoods of x. Moreover, since $\mathcal{T}$ and $\mathcal{V}(x)$
are closed under finite intersections, the same holds true
for $\mathcal{B}$. By (2.3), $\mathcal{B}$ generates a $\mathcal{Z}$-filter $\mathcal{Y}$. It follows
that $\mathcal{T} \subset \mathcal{Y}$ and $\mathcal{V}(x) \subset \mathcal{Y}$. Also, by (1) above the point x is a
limit point of $\mathcal{Y}$. Conversely, let $\mathcal{T} \subset \mathcal{Y}$ and let x be a
limit point of $\mathcal{Y}$. Then x is a cluster point of $\mathcal{Y}$ and hence
a cluster point of $\mathcal{T}$. This completes the proof.

A question which now naturally arises is whether or not
a $\mathcal{Z}$-filter has a <u>unique</u> limit point whenever a limit point
exists. As one might expect, the answer is affirmative for
Hausdorff spaces.

3.6 _Lemma._ _If the space X is Hausdorff, then_ $\mathfrak{X}$-_filters_
which converge have unique limit points. Moreover, if
$\mathfrak{X} \subset \mathbb{P}(X)$ _is a local base and if_ $\mathfrak{X}$-_filters which converge_
have unique limit points, then the space X is Hausdorff.

Proof: Let X be Hausdorff and let the points x and y
be distinct limit points of a $\mathfrak{X}$-filter $\mathfrak{T}$. Then there are
neighborhoods N(x) and N(y) of x and y, respectively,
satisfying $N(x) \cap N(y) = \emptyset$. There exist $F_1, F_2 \varepsilon \mathfrak{T}$ such that
$F_1 \subset N(x)$ and $F_2 \subset N(y)$. Therefore, the empty set belongs to
$\mathfrak{T}$ because $F_1 \cap F_2 \subset N(x) \cap N(y) = \emptyset$ which contradicts the finite
intersection property of $\mathfrak{T}$. Thus no $\mathfrak{X}$-filter can converge
to two distinct points.

Now, assume that $\mathfrak{X}$ is a local base and that $\mathfrak{X}$-filters
which converge have unique limit points. Suppose there are
two distinct points x and y in X satisfying $N(x) \cap N(y) \neq \emptyset$
for all neighborhoods of x and y. It follows that x is a
cluster point of the $\mathfrak{X}$-neighborhood filter $\mathfrak{v}(y)$. Hence, by
(3.5(2)) there is a $\mathfrak{X}$-filter $\mathfrak{T}$ such that x is a limit point
of $\mathfrak{T}$ and $\mathfrak{v}(y) \subset \mathfrak{T}$. However, from (3.5(1)) the point y is also
a limit point of $\mathfrak{T}$ which is a contradiction. Therefore there
are neighborhoods N(x) and N(y) of x and y, respectively, such
that $N(x) \cap N(y) = \emptyset$ which completes the proof.

Observe that for a Hausdorff space X, if a $\mathfrak{X}$-filter $\mathfrak{T}$
converges to a point $x \varepsilon X$, then x is the only cluster point of
$\mathfrak{T}$. Otherwise, some $\mathfrak{X}$-filter which contains $\mathfrak{T}$ would have
two distinct limit points. The following corollary relates
limit points and cluster points of $\mathfrak{X}$-ultrafilters. The proof
follows from (3.5(2)).

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3.7 <u>Corollary</u>. <u>Let</u> $\mathcal{B}$ <u>be a local base for a topological</u> <u>space</u> X <u>and let</u> $\mathcal{U}$ <u>be a</u> $\mathcal{B}$-<u>ultrafilter</u>. <u>A point</u> xεX <u>is a</u> <u>cluster point of</u> $\mathcal{U}$ <u>if and only if</u> x <u>is a limit point of</u> $\mathcal{U}$.

Another natural question to ask is whether or not two distinct $\mathcal{B}$-ultrafilters can have a common cluster point. The next two examples illustrate that different Bourbaki ultrafilters can indeed have common cluster points.

Let $\mathbb{N}^* = \mathbb{N} \cup \{ \infty \}$ denote the one-point compactification of the natural numbers $\mathbb{N}$. Let IE and ID represent the even and the odd integers respectively and let $\mathcal{B}$ be the power set of $\mathbb{N}^*$. It can be shown that the two collections

$\mathcal{T}_1 = \{ Z\varepsilon\mathcal{B} : (\mathbb{N}^* \smallsetminus Z) \cap IE \text{ is finite } \}$ and

$\mathcal{T}_2 = \{ Z\varepsilon\mathcal{B} : (\mathbb{N}^* \smallsetminus Z) \cap ID \text{ is finite } \}$ are both Bourbaki filters. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are contained in Bourbaki ultrafilters $\mathcal{U}_1$ and $\mathcal{U}_2$ respectively. Moreover, $\mathcal{U}_1 \neq \mathcal{U}_2$ because the collection of all even integers is a member of $\mathcal{U}_1$ while the collection of all odd integers is a member of $\mathcal{U}_2$. Furthermore, both $\mathcal{T}_1$ and $\mathcal{T}_2$ contain $\mathcal{V}(\infty)$. Hence, the point ∞ is a limit point, and so a cluster point, of both $\mathcal{U}_1$ and $\mathcal{U}_2$.

As another example, consider the real numbers under the indiscrete topology, let $\mathcal{B}$ be the power set of the reals, and let y be any real number. Then for each real number x, the Bourbaki ultrafilter $\mathcal{T}_x$ converges to y and hence has y as a cluster point.

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The next results reveal that the use of $\mathfrak{Z}$-filters for a suitable collection $\mathfrak{Z}$ eliminates the above situation. It is clear that the point x is a cluster point of the fixed $\mathfrak{Z}$-ultrafilter $\mathfrak{T}_x$ whenever $\mathfrak{T}_x$ exists (See (2.17)).

3.8 <u>Lemma</u>. <u>Let X be a topological space and let $\mathfrak{Z}$ be a disjunctive collection of closed subsets of X. The point x is a cluster point of the $\mathfrak{Z}$-filter $\mathfrak{T}$ if and only if $\mathfrak{T} \subset \mathfrak{T}_x$. Therefore, distinct $\mathfrak{Z}$-ultrafilters cannot have a common cluster point.</u>

<u>Proof</u>. Let x be a cluster point of $\mathfrak{T}$ and suppose that there is a $F \varepsilon \mathfrak{T}$ such that $F \notin \mathfrak{T}_x$. Then $x \notin F$ so that $X \backslash F$ is a neighborhood of x. It follows that $F \cap (X \backslash F) = \emptyset$ which contradicts the hypothesis that x is a cluster point of $\mathfrak{T}$. Therefore, $\mathfrak{T} \subset \mathfrak{T}_x$. Conversely, if $\mathfrak{T} \subset \mathfrak{T}_x$ then x is a cluster point of $\mathfrak{T}$ because x is a cluster point of $\mathfrak{T}_x$. Finally, if $\mathfrak{T}$ is a $\mathfrak{Z}$-ultrafilter with cluster point x then clearly $\mathfrak{T} = \mathfrak{T}_x$.

The proof of the following corollary consists of applying (3.3), (3.7) and (3.8).

3.9 <u>Corollary</u>. <u>If X is a topological space and $\mathfrak{Z}$ is a local base consisting of closed subsets of X, then $\mathfrak{T}_x$ is the unique $\mathfrak{Z}$-ultrafilter converging to the point x.</u>

When $\mathfrak{Z}$ is a disjunctive collection of closed sets, it is now clear that the map which takes each point $x \varepsilon X$ into the $\mathfrak{Z}$-ultrafilter $\mathfrak{T}_x$ is injective. After it becomes possible

to make $\omega(\mathfrak{Z})$ into a Hausdorff space, this map will be a
suitable candidate for embedding X into $\omega(\mathfrak{Z})$. Recall that
Wallman used the collection of all closed subsets of X for
his compactification. The fact that he could apply the normal
separation axiom to his distinguished collection enabled him
to establish a Hausdorff topology for the space of all
ultrafilters of closed subsets. By defining an analogous
property for the collection $\mathfrak{Z}$, the collection $\omega(\mathfrak{Z})$ of all
$\mathfrak{Z}$-ultrafilters can be made into a Hausdorff topological
space.

3.10 <u>Definition</u>. For a non-empty set X the collection
$\mathfrak{Z}$ is called <u>normal</u> in case for each pair $Z_1, Z_2 \epsilon \mathfrak{Z}$ satisfying
$Z_1 \cap Z_2 = \phi$ there exist $C_1, C_2 \epsilon \mathfrak{Z}$ such that $Z_1 \subset X \setminus C_1$, $Z_2 \subset X \setminus C_2$
and $(X \setminus C_1) \cap (X \setminus C_2) = \phi$.

It is easily seen that the power set of X is a normal
collection. <u>If</u> X <u>is a Tychonoff space, it can be shown that
the collection of all zero-sets is normal</u> ([11], 1.15). Also,
if X is a normal space, the collection of all closed subsets
of X is a normal collection.

The following lemma from ([33], 1.3) characterizes
$\mathfrak{Z}$-ultrafilters for normal collections.

3.11 <u>Lemma</u>. <u>If</u> $\mathfrak{Z}$ <u>is a normal collection and if</u> $\mathfrak{F}$ <u>is a</u>
$\mathfrak{Z}$<u>-filter, then the following statements are equivalent</u>:

(1) The $\mathfrak{Z}$-filter $\mathcal{T}$ <u>is a</u> $\mathfrak{Z}$-ultrafilter.

(2) <u>For each</u> $Z \epsilon \mathfrak{Z}$, $Z \cap F \neq \emptyset$ <u>for all</u> $F \epsilon \mathcal{T}$ <u>implies</u> <u>that</u> $Z \epsilon \mathcal{T}$.

(3) <u>For each</u> $Z \epsilon \mathfrak{Z}$, <u>either</u> $Z \epsilon \mathcal{T}$ <u>or there exists</u> <u>a</u> $Z_1 \epsilon \mathfrak{Z}$ <u>such that</u> $Z_1 \subset X \setminus Z$ <u>and</u> $Z_1 \epsilon \mathcal{T}$.

<u>Proof</u>: (1) implies (2). Let $Z \epsilon \mathfrak{Z}$ satisfying $Z \cap F \neq \emptyset$ for all $F \epsilon \mathcal{T}$. By (2.4), $\mathcal{T} \cup \{Z\}$ is contained in a $\mathfrak{Z}$-filter $\mathcal{Y}$. Since $\mathcal{T}$ is maximal it follows that $\mathcal{T} = \mathcal{T} \cup \{Z\} = \mathcal{Y}$. Hence, Z belongs to $\mathcal{T}$.

(2) implies (3). Let $Z \epsilon \mathfrak{Z}$ and $Z \notin \mathcal{T}$. By the hypothesis there is a $F \epsilon \mathcal{T}$ satisfying $Z \cap F = \emptyset$. Because $\mathfrak{Z}$ is normal, there are $C_1, C_2, \epsilon \mathfrak{Z}$ such that $Z \subset X \setminus C_1$, $F \subset X \setminus C_2$ and $(X \setminus C_1) \cap (X \setminus C_2) = \emptyset$. Then $F \subset C_1$ and so $C_1 \epsilon \mathcal{T}$. Furthermore, $C_1 \subset X \setminus Z$.

(3) implies (1). Suppose there is a $\mathfrak{Z}$-filter $\mathcal{Y}$ that properly contains $\mathcal{T}$. Then there is a $Z \epsilon \mathcal{Y}$ such that $Z \notin \mathcal{T}$. By hypothesis there exists a $Z_1 \epsilon \mathfrak{Z}$ satisfying $Z_1 \subset X \setminus Z$ and $Z_1 \epsilon \mathcal{T}$. It follows that $Z_1 \epsilon \mathcal{Y}$ and $Z \cap Z_1 = \emptyset$ which is a contradiction. Therefore, $\mathcal{T}$ is a $\mathfrak{Z}$-ultrafilter.

<u>Notice that</u> (3.11 (1) implies (2)) <u>does</u> <u>not</u> <u>require the</u> <u>normal property</u>. This observation will be useful later.

The theory that has been developed thus far in this manuscript has been achieved primarily by placing restrictions on the collection $\mathfrak{Z}$. These restrictions were largely moti-vated by properties of the power set, zero-sets in a Tychonoff space, or the closed sets in a normal space. Next, it will be helpful to consider certain types of $\mathfrak{Z}$-filters.

Let τ be a Bourbaki filter and let A and B be subsets

of X. It is easy to show that τ is a Bourbaki ultrafilter

if and only if whenever $A \cup B \varepsilon \tau$ then $A \varepsilon \tau$ or $B \varepsilon \tau$. However,

this result is not true for $\mathfrak{Z}$ -filters as the following

example illustrates. Let X be the natural numbers and let

$\mathfrak{Z}$ = { {1,2}, {2,3}, {2} , {1,2,3}, $\emptyset$} which is a ring of

sets. Then τ = { {2,3}, {1,2,3} } is a $\mathfrak{Z}$ -filter satisfying

$A \varepsilon \tau$ or $B \varepsilon \tau$ whenever $A \cup B \varepsilon \tau$. But τ is clearly not a

$\mathfrak{Z}$ -ultrafilter. These observations motivate the next

definition.

3.12 <u>Definition</u>. A $\mathfrak{Z}$ -filter τ is said to be <u>prime</u> in

case A, $B \varepsilon \mathfrak{Z}$ and $A \cup B \varepsilon \tau$ implies $A \varepsilon \tau$ or $B \varepsilon \tau$.

3.13 <u>Lemma</u>. <u>If</u> $\mathcal{U}$ <u>is a</u> $\mathfrak{Z}$-<u>ultrafilter, then</u> $\mathcal{U}$ <u>is prime</u>.

<u>Proof</u>: Let $A \cup B \varepsilon \mathcal{U}$ and $A \notin \mathcal{U}$. Then by (3.11 (1) implies

(2)) there is an $F \varepsilon \mathcal{U}$ such that $A \cap F = \emptyset$. It follows that

$B \cap F \neq \emptyset$ since $F \cap (A \cup B) \neq \emptyset$. Suppose $B \notin \mathcal{U}$. Then there is

a $Z \varepsilon \mathcal{U}$ satisfying $Z \cap B = \emptyset$. A contradiction will then be

reached because $(F \cap Z) \cap (A \cup B) = \emptyset$ would belong to $\mathcal{U}$. There-

fore, either $A \varepsilon \mathcal{U}$ or $B \varepsilon \mathcal{U}$.

3.14 <u>Lemma</u>. <u>If</u> $\mathfrak{Z}$ <u>is a normal collection that is a ring</u>

<u>of sets, then every prime</u> $\mathfrak{Z}$-<u>filter is embeddable in a unique</u>

$\mathfrak{Z}$ -<u>ultrafilter</u>.

<u>Proof</u>: Let τ be a prime $\mathfrak{Z}$ -filter. By (2.11) there is

a $\mathfrak{Z}$ -ultrafilter $\mathcal{U}$ containing τ. Now suppose τ is contained

in a $\mathfrak{Z}$ -ultrafilter $\mathcal{U}_1$ where $\mathcal{U} \neq \mathcal{U}_1$. Then there exist $A \varepsilon \mathcal{U}$ and

Bϵ $\mathcal{U}_1$ such that $A \cap B = \emptyset$ as a consequence of (3.11 (2)).
By the normality of $\mathfrak{Z}$ there exist $C_1, C_2 \epsilon \mathfrak{Z}$ satisfying $A \subset X \smallsetminus C_1$,
$B \subset X \smallsetminus C_2$ and $(X \smallsetminus C_1) \cap (X \smallsetminus C_2) = \emptyset$. It then follows that
$C_1 \cup C_2 = X$. Since $\mathfrak{Z}$ is a ring of sets $X\epsilon\mathfrak{Z}$ so that $C_1 \cup C_2 \epsilon \mathcal{T}$.
Without loss of generality, by the primeness of $\mathcal{T}$ assume
$C_1 \epsilon \mathcal{T}$. Then $C_1 \epsilon \mathcal{U}$ and since $A \subset X \smallsetminus C_1$ it follows that the
empty set is in $\mathcal{U}$ which is a contradiction and therefore
completes the proof.

The following lemma from ([33], 1.3) investigates the
convergence properties of prime $\mathfrak{Z}$-filters.

3.15 <u>Lemma</u>. <u>Let X be a</u> T_1<u>-topological space and let</u>
$\mathfrak{Z}$ <u>be a local base that is also a base for the closed subsets</u>
<u>of X. If xϵX and if</u> $\mathcal{T}$ <u>is a prime</u> $\mathfrak{Z}$<u>-filter on X, then the</u>
<u>following statements are equivalent</u>:

      (1) <u>The point x is a cluster point of</u> $\mathcal{T}$.

      (2) <u>The</u> $\mathfrak{Z}$<u>-filter</u> $\mathcal{T}$ <u>converges to x</u>.

      (3) $\cap\{ F : F \epsilon \mathcal{T}\} = \{ x \}$.

<u>Proof</u>: Let x be a cluster point of $\mathcal{T}$. Then $x \epsilon \cap\{F : F\epsilon \mathcal{T}\}$
because x must be in the closure of each $F\epsilon\mathcal{T}$ and the members
of $\mathcal{T}$ are closed sets. Now let $N(x)$ be a neighborhood of x.
By the local base property, there is a $Z\epsilon\mathfrak{Z}$ such that
$x\epsilon$ int $Z \subset Z \subset N(x)$. Hence the set $X\smallsetminus$int Z does not contain
the point x. Since $\mathfrak{Z}$ is a base for the closed sets, there
exists a $Z_1\epsilon\mathfrak{Z}$ satisfying $X\smallsetminus$int $Z \subset Z_1$ and $x \notin Z_1$. Hence
$x\epsilon X\smallsetminus Z_1 \subset$ int $Z \subset Z$. Therefore, $Z_1 \notin \mathcal{T}$. However, $Z_1 \cup Z = X$

belongs to $\mathcal{T}$ because $X \varepsilon \mathcal{B}$ whenever $\mathcal{B}$ is a base for the

closed sets. Hence, by the primeness of $\mathcal{T}$ the set Z belongs

to $\mathcal{T}$. Therefore $\mathcal{T}$ converges to x.

(2) implies (3). Let $\mathcal{T}$ converge to x. It has already

been observed that $x \varepsilon \cap \{F : F \varepsilon \mathcal{T}\}$. Suppose $y \varepsilon \cap \{F : F \varepsilon \mathcal{T}\}$

and $y \neq x$. Since X is a T_1-space, the set $X \setminus \{y\}$ is a

neighborhood of x. It follows that there is a $Z \varepsilon \mathcal{T}$ such

that $Z \subset X \setminus \{y\}$. The hypothesis that $y \varepsilon \cap \{F : F \varepsilon \mathcal{T}\}$ has

thus been contradicted.

The fact that (3) implies (1) is immediate.

The last result suggests the possibility that when a prime

$\mathcal{B}$-filter $\mathcal{T}$ converges to a point x then $\mathcal{T}$ is equal to the

$\mathcal{B}$-ultrafilter $\mathcal{T}_x$. The following example illustrates that

this is not necessarily the case.

Let X be the real numbers under the Euclidean topology

and let $\mathcal{B} = \{Z = cl\ Z \subset \mathbb{P}(X) : 0$ is not an accumulation

point of Z and on the boundary of Z $\}$. Notice that the only

closed intervals that do not belong to $\mathcal{B}$ are those which have

0 as an end point. The collection $\mathcal{B}$ is shown to be a ring

of sets as follows. Let $Z_1, Z_2 \varepsilon \mathcal{B}$. Now suppose 0 is an

accumulation point of $Z_1 \cap Z_2$. Then for every neighborhood

$N(0)$ of 0, $N(0) \cap ((Z_1 \cap Z_2) \setminus \{0\}) \neq \phi$ must hold. It follows

that 0 is an accumulation point of both Z_1 and Z_2 so that 0

is not on the boundary of either Z_1 or Z_2. Therefore, 0 is

not on the boundary of $Z_1 \cap Z_2$. Thus for any $Z_1, Z_2 \varepsilon \mathcal{B}$,

$Z_1 \cap Z_2 \varepsilon \mathcal{B}$. Next suppose 0 is an accumulation point of

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$Z_1 \cup Z_2$ where $Z_1, Z_2 \epsilon \, \mathfrak{Z}$. In the following argument it is
sufficient to consider only open intervale about 0 since
the open intervals form a base for the open sets. If 0 is
an accumulation point of Z_1, then $0 \, \epsilon$ int Z_1 by the way the
$\mathfrak{Z}$ sets are defined. In this case, 0 is not on the boundary
of $Z_1 \cup Z_2$. On the other hand, if 0 is not an accumulation
point of Z_1 then there is an open interval $N(0)$ about 0 such
that $N(0) \cap (Z_1 \smallsetminus \{0\}) = \emptyset$. It follows that 0 is an accumula-
tion point of Z_2 and 0 is not on the boundary of $Z_1 \cup Z_2$.
Hence for any $Z_1, Z_2 \epsilon \, \mathfrak{Z}$, $Z_1 \cup Z_2 \epsilon \, \mathfrak{Z}$.

Moreover, the following argument shows that $\mathfrak{Z}$ is a local
base. To see this, observe that for any point x and any open
interval $N(x) = (x-a, x+b)$ about x, the closed interval
$Z = [x-a/2, x+b/2]$ belongs to $\mathfrak{Z}$ unless $x-a/2 = 0$ or
$x-b/2 = 0$ and $x \epsilon$ int $Z \subset Z \subset N(x)$. In case an endpoint is
equal to 0, then $Z_1 = [x-a/3, x+b/3]$ belongs to $\mathfrak{Z}$ and $\mathfrak{z}$
$x \epsilon$ int $Z_1 \subset Z_1 \subset N(x)$. Therefore, $\mathfrak{Z}$ is a local base.

Next, it will be shown that $\mathfrak{Z}$ is a base for the closed
sets. Since any closed set F is the union of a collection of
non-overlapping closed intervals, F belongs to $\mathfrak{Z}$ provided 0
is not an end point of one of these closed intervals. Suppose
F is the union of a collection $\mathfrak{C}$ of non-overlapping closed
intervals where $[0,b]$ belongs to this collection. Then
$\cup \{ C : C \epsilon \, \mathfrak{C} \} \smallsetminus [0,b] \epsilon \, \mathfrak{Z}$ and, since $[-1/n, b] \epsilon \, \mathfrak{Z}$ for each
$n \epsilon \, \mathbb{N}$ and $[0,b] = \cap \{ [-1/n, b] : n \epsilon \, \mathbb{N} \}$,
$\cap \{ (\cup \{ C : C \epsilon \, \mathfrak{C} \} \smallsetminus [0,b]) \cup [-1/n, b] : n \epsilon \, \mathbb{N} \} = F$. Hence,
$\mathfrak{Z}$ is a base for the closed sets.

Furthermore, the collection $\mathfrak{C} = \{ Z \epsilon \mathfrak{Z} : 0 \epsilon \text{ int } Z \}$ is a prime $\mathfrak{Z}$-filter converging to 0. To see that $\mathfrak{C}$ is prime, let $A, B \epsilon \mathfrak{Z}$ with $A \cup B \epsilon \mathfrak{C}$. Hence, $0 \epsilon \text{ int } (A \cup B)$. Without loss of generality, let $0 \epsilon A$. If $0 \epsilon \text{ int } A$, then $A \epsilon \mathfrak{C}$. So suppose $0 \notin \text{ int } A$. Then 0 can not be an accumulation point of A by the defining properties of $\mathfrak{Z}$. Therefore, there is a neighborhood $N(0)$ of 0 disjoint from $A \smallsetminus \{0\}$. Also, there is a neighborhood $N_1(0)$ of 0 which is contained in $A \cup B$ because $0 \epsilon \text{ int } (A \cup B)$. Then $N(0) \cap N_1(0) \subseteq A \cup B$. It follows that $N(0) \cap N_1(0) \subseteq B$ so that $0 \epsilon \text{ int } B$ and $B \epsilon \mathfrak{C}$. Hence $\mathfrak{C}$ is prime.

However, $\mathfrak{C} \cup \{ 0 \}$ clearly generates a $\mathfrak{Z}$-filter that properly contains $\mathfrak{C}$.

The clusterable $\mathfrak{Z}$-filters form another special class of $\mathfrak{Z}$-filters. A $\mathfrak{Z}$-filter $\mathfrak{C}$ on a topological space X is called clusterable in case $\mathfrak{C}$ converges to each of its cluster points. This notion has been investigated in ([33], 1.3) and is useful in the study of compactness for uniform spaces. However, there is no special theory concerning these $\mathfrak{Z}$-filters and so only their existence is mentioned here. Observe that every prime $\mathfrak{Z}$-filter is clusterable.

Generally, the intersection of a collection of members of a $\mathfrak{Z}$-filter $\mathfrak{C}$ does not belong to $\mathfrak{C}$ when there are more than finitely many elements in the collection. For instance, the proof that $\mathfrak{Z}$ was a base for the closed sets in the last example revealed that the intersection of a countable

collection of members of $\mathcal{T}$ need not even be an element of $\mathcal{B}$.
In fact, the intersection of countably many elements of
$\mathcal{T}$ may be the empty set. Let X be the natural numbers and
consider the Bourbaki filter $\mathcal{T} = \{ Z \subset \mathbb{P}(X) : X \setminus Z$ is
finite $\}$. Then $X \setminus \{n\}$ belongs to $\mathcal{T}$ for each $n \in \mathbb{N}$ and
$\cap \{ X \setminus \{n\} \varepsilon \mathcal{T} : n \varepsilon \mathbb{N} \} = \phi$.

3.16 <u>Definition</u>. A $\mathcal{B}$-filter $\mathcal{T}$ is said to have the
<u>countable</u> <u>intersection</u> <u>property</u> in case the intersection of
every countable collection of members of $\mathcal{T}$ is non-empty. The
$\mathcal{B}$-filter $\mathcal{T}$ is said to be <u>closed</u> <u>under</u> <u>countable</u> <u>intersections</u>
in case the intersection of every countable collection of
members of $\mathcal{T}$ belongs to $\mathcal{T}$.

Recall that in Section I realcompact spaces were defined.
The following equivalent formulation is more useful in con-
structing realcompactifications. The proof is omitted (See
[33], 2.5.1).

3.17 <u>Lemma</u>. <u>A</u> <u>Tychonoff</u> <u>space</u> X <u>is</u> <u>realcompact</u> <u>if</u> <u>and</u>
<u>only</u> <u>if</u> <u>every</u> <u>zero-set-ultrafilter</u> <u>on</u> X <u>with</u> <u>the</u> <u>countable</u>
<u>intersection</u> <u>property</u> <u>is</u> <u>fixed</u>.

Observe from (2.15) that every compact space is realcompact.
When $\mathcal{B}$ is a local base, the $\mathcal{B}$-neighborhood filter $\mathcal{V}(x)$
provides an example of a $\mathcal{B}$-filter with the countable inter-
section property that is not closed under countable inter-
sections. However, the following lemma proves the equivalence
of these concepts under certain conditions.

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3.18 Lemma. If $\mathfrak{Z}$ is closed under countable intersections, then a $\mathfrak{Z}$-ultrafilter has the countable intersection property if and only if it is closed under countable intersections.

Proof: Let $\mathcal{U}$ be a $\mathfrak{Z}$-ultrafilter with the countable intersection property. Let $\mathcal{G}$ be a countable collection of members of $\mathcal{U}$ and denote the intersection of the members of $\mathcal{G}$ by Z. For any $A\varepsilon\mathcal{U}$ the collection $\mathcal{G}\cup\{A\}$ is still countable so that $Z\cap A\neq\phi$. By (2.4) the collection $\mathcal{U}\cup\{Z\}$ generates a $\mathfrak{Z}$-filter $\mathcal{E}$. It follows that $\mathcal{U}=\mathcal{E}$ and $Z\varepsilon\mathcal{E}=\mathcal{U}$. Hence $\mathcal{U}$ is closed under countable intersections. It is clear that if $\mathcal{U}$ is closed under countable intersections then it has the countable intersection property because the empty set is not in $\mathcal{U}$.

The proof of the next lemma, which provides an example of a realcompact space, is omitted. It may be found in ([33], 2.5.2).

3.19 Lemma. A Tychonoff space X is Lindelöf if and only if every zero-set-filter ($\mathcal{Z}$-filter) on X with the countable intersection property is fixed.

Hence, when a space X is Lindelöf every $\mathcal{Z}$-filter, and in particular every $\mathcal{Z}$-ultrafilter, with the countable inter- section property is fixed and so X is realcompact by (3.17). Hence, every second countable Tychonoff space is realcompact as is every separable metric space. In fact, every subspace of a Euclidean space is realcompact.

In view of the above results a natural question to ask is whether or not an arbitrary $\aleph$-filter with the countable intersection property can be embedded in a $\aleph$-ultrafilter with the countable intersection property. An example where such an embedding cannot take place is given in ([33], 2.8). Briefly, the set of all real numbers X is made into a topological space E'_μ by defining a base for the open sets to be the intervals of the form $(a,b] = \{ x \varepsilon X : a < x \leq b \}$. It is shown that the product space $E'_\mu \times E'_\mu$ is realcompact but not Lindelöf. Suppose every $\mathcal{Z}$-filter with the countable intersection property is embeddable in a $\mathcal{Z}$-ultrafilter with the countable intersection property. Since $E'_\mu \times E'_\mu$ is realcompact, it would follow that every $\mathcal{Z}$-filter with the countable intersection property is fixed. Then by (3.19), $E'_\mu \times E'_\mu$ is Lindelöf which is a contradiction. Hence, such an embedding is not generally possible.

The introduction of the following concepts will allow for such an embedding in the case of prime $\aleph$-filters. Moreover, the collection of zero-sets on a Tychonoff space enjoys these properties. Both ideas will be incorporated into a stronger notion which turns out to be useful in the Wallman-type realcompactification.

3.20 <u>Definition</u>. A non-empty collection $\aleph$ is said to be a <u>delta</u> <u>ring</u> <u>of</u> <u>sets</u> in case $\aleph$ is a ring of sets that is closed under countable intersections. The collection $\aleph$ is said to be <u>complement</u> <u>generated</u> in case for each $Z \varepsilon \aleph$ there exists a sequence $\{ C_n : n \varepsilon N \}$ of complements of members of $\aleph$ satisfying $Z = \cap \{ C_n : n \varepsilon N \}$.

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The following lemma is due to M. Weir and is Theorem
3.16 of ([33], 1.3).

3.21 $\underline{Lemma}$. $\underline{If}$ $\mathfrak{Z}$ $\underline{is}$ $\underline{a}$ $\underline{delta}$ $\underline{ring}$ $\underline{of}$ $\underline{sets}$ $\underline{on}$ X $\underline{that}$ $\underline{is}$
$\underline{normal}$ $\underline{and}$ $\underline{complement}$ $\underline{generated}$, $\underline{then}$ $\underline{every}$ $\underline{prime}$ $\mathfrak{Z}$-$\underline{filter}$
$\underline{with}$ $\underline{the}$ $\underline{countable}$ $\underline{intersection}$ $\underline{property}$ $\underline{is}$ $\underline{embeddable}$ $\underline{in}$ $\underline{a}$
$\underline{unique}$ $\mathfrak{Z}$-$\underline{ultrafilter}$ $\underline{with}$ $\underline{the}$ $\underline{countable}$ $\underline{intersection}$
$\underline{property}$.

$\underline{Proof}$: Let $\mathfrak{T}$ be a prime $\mathfrak{Z}$-filter on X with the countable
intersection property. By (2.11) and (3.14), $\mathfrak{T}$ is contained
in a unique $\mathfrak{Z}$-ultrafilter $\mathfrak{U}$. Suppose there exists a
countable collection $\{ Z_n : n \varepsilon \mathbb{N} \}$ of members of $\mathfrak{U}$ which has
empty intersection. Then the family $\{ X \setminus Z_n : n \varepsilon \mathbb{N} \}$ is a
countable cover of X. Since $\mathfrak{Z}$ is complement generated, for
each $n \varepsilon \mathbb{N}$ there exists a sequence $\{ C_{n,i} : i \varepsilon \mathbb{N} \}$ of complements
of members of $\mathfrak{Z}$ such that $Z_n = \cap \{ C_{n,i} : i \varepsilon \mathbb{N} \}$. It follows
that $X = \cup \{ X \setminus Z_n : n \varepsilon \mathbb{N} \} = \cup_{n \varepsilon \mathbb{N}} \cup_{i \varepsilon \mathbb{N}} Z'_{n,i}$ where each $Z'_{n,i} \varepsilon \mathfrak{Z}$
and $X \setminus C_{n,i} = Z'_{n,i} \subset X \setminus Z_n$. Then (after suitable re-labeling)
the countable collection $\{ Z'_n : n \varepsilon \mathbb{N} \}$ covers X. For each
$n \varepsilon \mathbb{N}$, it is possible to select an index i_n such that
$Z'_n \subset X \setminus Z_{i_n}$. Since $\mathfrak{Z}$ is normal, there exist $\overline{Z}_1, \overline{Z}_2 \varepsilon \mathfrak{Z}$ satis-
fying $Z'_n \subset X \setminus \overline{Z}_1$, $Z_{i_n} \subset X \setminus \overline{Z}_2$ and $(X \setminus \overline{Z}_1) \cap (X \setminus \overline{Z}_2) = \emptyset$. It
follows that $\overline{Z}_1 \cup \overline{Z}_2 = X$ and that each of $\overline{Z}_1$ and $\overline{Z}_2$ can meet
only one of the sets Z'_n and Z_{i_n}. Since $\mathfrak{Z}$ is a ring of sets,
it also follows that $X \varepsilon \mathfrak{Z}$. Hence $X \varepsilon \mathfrak{T}$ so that by the primeness
of $\mathfrak{T}$, $B_n \varepsilon \mathfrak{T}$ where $B_n = \overline{Z}_1$ or $B_n = \overline{Z}_2$. Moreover, $B_n \varepsilon \mathfrak{U}$ so that

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$B_n \cap Z_{i_n} \neq \emptyset$.   Hence $B_n \cap Z_n' = \emptyset$

must hold. Therefore $\cap\{ B_n : n \in \mathbb{N} \} \subset \cap\{ X \setminus Z_n' : n \in \mathbb{N}\} = \emptyset$.

The countable intersection property of $\mathcal{T}$ has thus been

contradicted. Therefore, there does not exist a countable

subcollection of $\mathcal{U}$ with empty intersection.

Notice that (3.21) is valid for the collection of zero-

sets on a Tychonoff space.

In studying the problem associated with obtaining

Hausdorff compactifications of an arbitrary Tychonoff space,

Frink [10] generalized Wallman's method by using the following

concept.

3.22 <u>Definition</u>. Let X be a topological space. The

collection $\mathcal{Z}$ is a <u>normal base</u> on X in case $\mathcal{Z}$ is a ring of

sets that is disjunctive, normal, and a base for the closed

sets.

It is clear that in a Tychonoff space X the collection

$\mathcal{Z}$ (X) of all zero-sets is a normal base.

For a normal base $\mathcal{Z}$ on a Tychonoff space X, Frink made

the collection $\omega(\mathcal{Z})$ of all $\mathcal{Z}$-ultrafilters into a topological

space in the following way. The collection of all sets of the

form $Z^\omega = \{ \mathcal{T} \in \omega(\mathcal{Z}) : Z \in \mathcal{T}\}$ for $Z \in \mathcal{Z}$ is taken as a base for the

closed sets. To see that these sets form a base for the

closed sets, it is sufficient to show the collection is

closed under finite unions. If $\mathcal{T} \in (Z_1^\omega \cup Z_2^\omega)$, then $Z_1 \in \mathcal{T}$ or

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$Z_2 \varepsilon \, \mathfrak{T}$. It follows that $Z_1 \cup Z_2 \varepsilon \, \mathfrak{T}$ so that $\mathfrak{T} \varepsilon (Z_1 \cup Z_2)^\omega$. By

(3.13) the $\mathfrak{Z}$-ultrafilter $\mathfrak{T}$ is prime so that the argument is

reversible. Therefore, $Z_1^\omega \cup Z_2^\omega = (Z_1 \cup Z_2)^\omega$.

Because $\mathfrak{Z}$ is a disjunctive collection of closed subsets

of X, from (3.8) it follows that the $\mathfrak{Z}$-filter

$\mathfrak{T}_x = \{\, Z \varepsilon \, \mathfrak{Z} : x \varepsilon Z \,\}$ is the unique $\mathfrak{Z}$-ultrafilter converging

to the point x. Hence, the mapping f from X into $\omega(\mathfrak{Z})$

defined by $f(x) = \mathfrak{T}_x$ is injective. Furthermore, f is a

__homeomorphism__ from X onto f[X]. For let $Z \varepsilon \, \mathfrak{Z}$ be a basic

closed set. Then $\mathfrak{T} \varepsilon f[Z]$ if and only if $\mathfrak{T} = \mathfrak{T}_x$ for some $x \varepsilon Z$.

It follows that $Z \varepsilon \, \mathfrak{T}_x$ so that $\mathfrak{T}_x \varepsilon Z^\omega$. Hence,

$\mathfrak{T} \varepsilon f[X] \cap Z^\omega$. The argument is reversible so that

$f[Z] = f[X] \cap Z^\omega$.

The next argument proves that every non-empty basic open

set in $\omega(\mathfrak{Z})$ meets f[X] from which it may be concluded that

f[X] is __dense__ in $\omega(\mathfrak{Z})$. A non-empty, basic open set of $\omega(\mathfrak{Z})$

is of the form $U^\omega = \{\, \mathfrak{T} \varepsilon \omega(\mathfrak{Z}) : \text{there exists } A \varepsilon \, \mathfrak{T} \text{ such that}$

$A \subset U$ and $(X \smallsetminus U) \varepsilon \, \mathfrak{Z} \,\}$. In a manner similar to that of the

previous paragraph it follows that $f[U] = f[X] \cap U^\omega$ for every

open set U such that $X \smallsetminus U \varepsilon \, \mathfrak{Z}$. Moreover, if U^ω is non-empty,

then f[U] is non-empty and hence $f[X] \cap U^\omega$ is also non-empty.

Furthermore, the space $\omega(\mathfrak{Z})$ is __Hausdorff__. Let $\mathfrak{T}_1$ and

$\mathfrak{T}_2$ be distinct $\mathfrak{Z}$-ultrafilters. Then by (3.11(2)), there

exist $Z_1 \varepsilon \, \mathfrak{T}_1$ and $Z_2 \varepsilon \, \mathfrak{T}_2$ satisfying $Z_1 \cap Z_2 = \phi$. Since $\mathfrak{Z}$ is

a normal collection, there are sets C_1, $C_2 \varepsilon \, \mathfrak{Z}$ such that

$Z_1 \subset X \smallsetminus C_1$, $Z_2 \subset X \smallsetminus C_2$ and $(X \smallsetminus C_1) \cap (X \smallsetminus C_2) = \phi$. Hence,

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$\mathfrak{T}_1 \varepsilon (X \smallsetminus C_1)^\omega$, $\mathfrak{T}_2 \varepsilon (X \smallsetminus C_2)^\omega$ and $(X \smallsetminus C_1)^\omega \cap (X \smallsetminus C_2)^\omega = \phi$.

Finally, the space $\omega(\mathfrak{Z})$ is <u>compact</u>. For let ϕ^ω be a collection of closed sets in $\omega(\mathfrak{Z})$ with the finite inter-section property. It is sufficient to consider ϕ^ω as a collection of basic closed sets. Let $\phi = \{ Z\varepsilon\mathfrak{Z}: Z^\omega \varepsilon\phi^\omega\}$. It is easily seen that ϕ has the finite intersection property. Therefore, it follows that there is a $\mathfrak{Z}$-ultrafilter $\mathfrak{T}$ such that $\phi \subset \mathfrak{T}$. If $Z\varepsilon\phi$, then $Z\varepsilon\mathfrak{T}$ so that $\mathfrak{T}\varepsilon Z^\omega$. It follows that $\mathfrak{T}$ belongs to the intersection of the members of ϕ^ω.

It has thus been established that $\omega(\mathfrak{Z})$ is a compact Hausdorff space that contains a dense homeomorphic copy of X. Furthermore, Frink showed that when $\mathfrak{Z}$ is the collection $\mathcal{Z}$ (X) of all zero-sets, the space $\omega(\mathfrak{Z})$ is precisely the Stone-Čech compactification βX (within a homeomorphism). It can be shown that when $\mathfrak{Z}$ is the collection of zero-sets of those continuous functions on X that are constant on the complement of some compact subset of X, the space $\omega(\mathfrak{Z})$ is the Alexandroff one-point compactification [1].

The normal base concept plays another important role in the study of topological spaces because it furnishes an inter-nal characterization of completely regular T_1-spaces. <u>A</u> T_1-<u>topological space is completely regular if and only if it has a normal base</u>. The necessity follows from the fact that the zero-sets are a normal base for a Tychonoff space. Conversely, when a T_1-space has a normal base, then it has a Frink compacti-fication and is thus completely regular.

Alo and Shapiro introduced the following concept which generalizes the normal base property and is useful in constructing Wallman-type realcompactifications of a Tychonoff space.

3.23 Definition. Let X be a topological space. A collection $\mathfrak{Z}$ is a strong delta normal base on X in case $\mathfrak{Z}$ is a delta ring of sets that is a normal base and complement generated.

From previous results and remarks, it is clear that the collection $\mathbb{Z}(X)$ of all zero-sets is a strong delta normal base when X is a Tychonoff space. In fact, it can be shown that every strong delta normal base is a subcollection of the zero-sets ([33], 2.7.8).

For a strong delta normal base $\mathfrak{Z}$ on a Tychonoff space X, Alo and Shapiro considered the collection $\rho(\mathfrak{Z}) = \{\mathfrak{Z}\epsilon\omega(\mathfrak{Z}) : \mathfrak{Z}$ has the countable intersection property$\}$. Moreover, they showed that $\rho(\mathfrak{Z})$ considered as a subspace of $\omega(\mathfrak{Z})$ is a realcompact space that contains a dense homeomorphic copy of X. They also showed that if $\mathfrak{Z}$ is the collection of all zero-sets, then $\rho(\mathfrak{Z})$ is precisely the Hewitt realcompactification υX. Furthermore, every G_δ - set in $\rho(\mathfrak{Z})$ meets the homeomorphic image of X. Their work may be found in [3].

In the above work, Alo and Shapiro also give an example to show that different strong delta normal bases may produce different realcompactifications. An open question is whether

or not every realcompactification is of the form $\rho(\mathfrak{Z})$ for some strong delta normal base or $\omega(\mathfrak{Z})$ for some normal base (since compactifications are realcompactifications). E. Steiner has studied this question in [28] and introduced the following concept.

3.24 <u>Definition</u>. Let X be a non-empty set. The collection $\mathfrak{Z}$ is said to be <u>nest</u> <u>generated</u> if for each $Z \epsilon \mathfrak{Z}$ there exists a sequence $\{ Z_n \epsilon \mathfrak{Z} : n \epsilon \mathbb{N} \}$ of elements of $\mathfrak{Z}$ and a sequence $\{ C_n : n \epsilon \mathbb{N} \}$ of complements of members of $\mathfrak{Z}$ such that $Z = \cap \{ Z_n : n \epsilon \mathbb{N} \}$ and $C_{n+1} \subset Z_{n+1} \subset C_n \subset Z_n$ for each $n \epsilon \mathbb{N}$.

The next result is due to Weir [4].

3.25 <u>Lemma</u>. <u>Let</u> X <u>be</u> <u>a</u> <u>non-empty</u> <u>set</u>. <u>The</u> <u>collection</u> $\mathfrak{Z}$ <u>is</u> <u>a</u> <u>nest</u> <u>generated</u> <u>delta</u> <u>ring</u> <u>if</u> <u>and</u> <u>only</u> <u>if</u> $\mathfrak{Z}$ <u>is</u> <u>a</u> <u>normal</u> <u>complement</u> <u>generated</u> <u>delta</u> <u>ring</u>.

<u>Proof</u>: Let $\mathfrak{Z}$ be nest generated and let $A, B \epsilon \mathfrak{Z}$ with $A \cap B = \emptyset$. Then there are sequences $\{ A_n : n \epsilon \mathbb{N} \}$ and $\{ B_n : n \epsilon \mathbb{N} \}$ in $\mathfrak{Z}$ whose intersections are A and B respectively. Also there are sequences $\{ X \smallsetminus C_n : n \epsilon \mathbb{N} \}$ and $\{ X \smallsetminus D_n : n \epsilon \mathbb{N} \}$ of complements of $\mathfrak{Z}$ sets such that $X \smallsetminus C_{n+1} \subset A_{n+1} \subset X \smallsetminus C_n \subset A_n$ and $X \smallsetminus D_{n+1} \subset B_{n+1} \subset X \smallsetminus D_n \subset B_n$ for each $n \epsilon \mathbb{N}$. Let $P = \cup \{ (X \smallsetminus C_n) \cap (X \smallsetminus B_n) : n \epsilon \mathbb{N} \}$ and $Q = \cup \{ (X \smallsetminus D_n) \cap (X \smallsetminus A_n) : n \epsilon \mathbb{N} \}$. These are disjoint complements of members of $\mathfrak{Z}$. For suppose $x \epsilon P$. Then $x \epsilon (X \smallsetminus C_n)$ and $x \epsilon (X \smallsetminus B_n)$ for some n. It follows that if $x \epsilon (X \smallsetminus A_m)$ then $m < n$ while $x \epsilon X \smallsetminus D_m$ only if $m > n$. Hence $x \epsilon Q$. A similar argument

holds for $x \in Q$. By re-writing P and Q as $P = X \setminus \cap \{ C_n \cup B_n : n \in \mathbb{N} \}$
and $Q = X \setminus \cap \{ D_n \cup A_n : n \in \mathbb{N} \}$, it is clear that they are
complements of $\mathfrak{Z}$ sets. Furthermore $A \subset P$ and $B \subset Q$. For
suppose $x \in A$. Since $A \cap B = \emptyset$ and $B = \{ B_n : n \in \mathbb{N} \}$, there is
an n such that $x \in X \setminus B_n$. It is clear that $x \in X \setminus C_n$. Hence
$x \in (X \setminus A_n) \cap (X \setminus B_n)$ so $x \in P$. Similarly for Q. Therefore $\mathfrak{Z}$ is
normal. Finally let $Z \in \mathfrak{Z}$. Then there exists a sequence
$\{ A_n : n \in \mathbb{N} \}$ in $\mathfrak{Z}$ and a sequence $\{ C_n : n \in \mathbb{N} \}$ of complements
of $\mathfrak{Z}$ sets which satisfy the nest generating property. From
the facts that $Z = \cap \{ A_n : n \in \mathbb{N} \}$ and the nest generating
property, it is clear that $Z = \cap \{ C_n : n \in \mathbb{N} \}$. Hence $\mathfrak{Z}$ is
complement generated.

Conversely, let $\mathfrak{Z}$ be a normal, complement generated delta
ring. Let $Z \in \mathfrak{Z}$. Since $\mathfrak{Z}$ is complement generated,
$Z = \cap \{ X \setminus Z_n : n \in \mathbb{N} \}$ where each $Z_n \in \mathfrak{Z}$. Since $Z \cap Z_1 = \emptyset$, by
the normality of $\mathfrak{Z}$ there exist sets $A_1, B_1 \in \mathfrak{Z}$ such that
$Z \subset X \setminus A_1$, $Z_1 \subset X \setminus B_1$ and $(X \setminus A_1) \cap (X \setminus B_1) = \emptyset$. Hence
$Z \subset X \setminus A_1 \subset B_1 \subset X \setminus Z_1$. Define A_1' by $X \setminus A_1' = X \setminus (A_1 \cup Z_2)$. Then
$A_1' \in \mathfrak{Z}$. Moreover, $(X \setminus A_1') \cap Z_2 = \emptyset$ and $Z \subset X \setminus A_1' \subset B_1 \subset X \setminus Z_1$ as
is easily versified. Next there exist sets $A_2, B_2 \in \mathfrak{Z}$ such that
$Z \subset X \setminus A_2$, $A_1' \subset X \setminus B_2$ and $(X \setminus A_2) \cap (X \setminus B_2) = \emptyset$. Hence,
$Z \subset X \setminus A_2 \subset B_2 \subset X \setminus A_1' \subset B_1$. Moreover $B_2 \subset X \setminus Z_2$ since
$(X \setminus A_1') \cap Z_2 = \emptyset$. Next define A_2' by $X \setminus A_2' = X \setminus (A_2 \cup Z_3)$.
Again $A_2' \in \mathfrak{Z}$ and $(X \setminus A_2') \cap Z_3 = \emptyset$. Also $Z \subset X \setminus A_2' \subset B_2 \subset X \setminus A_1' \subset B_1$.

By induction define sequences $\{ A_n' : n \in \mathbb{N} \}$ and
$\{ B_n : n \in \mathbb{N} \}$ in $\mathfrak{Z}$ as follows: Assume that for $i = 1, \cdots, k$

the sets A_i' and B_i have been defined such that

    (1) $B_i \subset X \setminus Z_i$

    (2) $(X \setminus A_i) \cap Z_{i+1} = \phi$.

    (3) $Z \subset X \setminus A_i' \subset B_i \subset X \setminus A_{i-1}' \subset B_{i-1}$

Then since $Z \cap A_k' = \phi$, there exist sets A_{k+1} and B_{k+1} such

that $Z \subset X \setminus A_{k+1}$, $A_k' \subset X \setminus B_{k+1}$ and $(X \setminus A_{k+1}) \cap (X \setminus B_{k+1}) = \phi$.

Thus $Z \subset X \setminus A_{k+1} \subset B_{k+1} \subset X \setminus A_k' \subset B_k$. Define A_{k+1}' by $X \setminus A_{k+1}' =$

$X \setminus (A_{k+1} \cup Z_{k+2})$. Then $A_{k+1}' \epsilon \mathcal{B}$ and $Z \subset X \setminus A_{k+1}' \subset B_{k+1} \subset X \setminus A_k' \subset B_k$.

Also, $(X \setminus A_{k+1}') \cap Z_{k+2} = \phi$ and $B_{k+1} \subset X \setminus A_k'$. It then follows

that $Z = \cap \{ B_n : n \epsilon \mathbb{N} \}$ from the fact that $B_n \subset X \setminus Z_n$ for

every $n \epsilon \mathbb{N}$. Hence, $\mathcal{B}$ is nest generated.

3.25(1) Remark. E. Steiner defined a collection $\mathcal{B}$ of

closed sets as <u>separating</u> in case for any $x \epsilon X$ and any closed

set F such that $x \notin F$ there exists Z_1, $Z_2 \epsilon \mathcal{B}$ satisfying $x \epsilon Z_1$,

$F \subset Z_2$ and $Z_1 \cap Z_2 = \phi$. Actually, the preceding result (3.25)

states that <u>a collection</u> $\mathcal{B}$ <u>of closed sets is a separating nest</u>

<u>generated delta ring if and only if</u> $\mathcal{B}$ <u>is a strong delta</u>

<u>normal base</u>.

After (3.25) a natural question to ask is whether or not

a complement generated delta ring is nest generated. The

following example answers this question negatively.

Let X be the natural numbers and let $\mathcal{B} = \{ Z \subset \mathbb{P}(X) : Z$

has finitely many points $\}$. Clearly $\mathcal{B}$ is a delta ring. Let

$Z \epsilon \mathcal{B}$ such that $Z = \{ n_1, n_2, \cdots n_k \}$. Define a sequence as

follows: let $Z_1 = \{ \min [X \setminus Z] \}$, $Z_2 = \{ \min [Z \setminus (Z \cup Z_1) \}$,

$\cdots Z_i = \{ \min [X \setminus (Z \cup Z_1 \cdots \cup Z_{i-1}] \} \cdots$. Each Z_i consists

of a single point. Hence $Z_i \epsilon \mathfrak{Z}$ for each i. Also

$Z = \cap \{ X \setminus Z_i : i \epsilon \mathbb{N} \}$ for suppose $n \epsilon Z$. Then $n \notin Z_i$ for each i

so that $n \epsilon \cap \{ X \setminus Z_i : i \epsilon \mathbb{N} \}$. On the other hand suppose

$n \epsilon \cap \{ X \setminus Z_i : i \epsilon \mathbb{N} \}$. If $n \notin Z$ eventually $\{ n \} = Z_m$ some m.

Hence $n \notin X \setminus Z_m$. This would lead to a contradiction. Thus $n \epsilon Z$

and equality has been proven. However $\mathfrak{Z}$ is not nest generated.

For suppose it is. Then $C_{n+1} \subset Z_{n+1} \subset C_n \subset Z_n$ where $Z_{n+1} \epsilon \mathfrak{Z}$ and

thus contains finitely many points. But C_{n+1} is the comple-

ment of a $\mathfrak{Z}$ set and thus has infinitely many points with

$C_{n+1} \subset Z_{n+1}$ which is impossible.

Except for a few results on tracing when $\mathfrak{Z}$-filters were

first introduced, this concept has not been explored in this

thesis. The remainder of this section will concentrate on

this area

3.26 <u>Lemma</u>. <u>Let</u> $A \subset X$ <u>be a non-empty subset of the set</u>

X <u>such that</u> $\mathfrak{Z} \cap A \neq \emptyset$. <u>The following statements are true.</u>

(1) <u>If</u> $\mathfrak{Z}$ <u>is</u> $\mathfrak{Z}$-<u>disjunctive then</u> $\mathfrak{Z} \cap A$ <u>is</u>

$\mathfrak{Z} \cap$ A-<u>disjunctive.</u>

(2) <u>If</u> $\mathfrak{Z}$ <u>is a delta ring of sets then</u> $\mathfrak{Z} \cap A$ <u>is a</u>

<u>delta ring of sets.</u>

(3) <u>If</u> $\mathfrak{Z}$ <u>is complement generated then</u> $\mathfrak{Z} \cap A$ <u>is comple-</u>

<u>ment generated.</u>

<u>Proof</u>: (1) Let $Z_A \epsilon \mathfrak{Z} \cap A$ and $x \epsilon A \setminus Z_A$. Then $x \notin Z$ where

$Z \cap A = Z_A$. Since $\mathfrak{Z}$ is $\mathfrak{Z}$-disjunctive, there is a $Z \epsilon \mathfrak{Z}$ such

that $x\epsilon Z'$ and $Z' \cap Z = \phi$. Then $x\epsilon Z'_A = Z' \cap A$ and $Z'_A \cap Z_A = \phi$.
Hence $\mathcal{Z} \cap A$ is $\mathcal{Z} \cap A$-disjunctive.

(2)   Let $Z_A, Z'_A \epsilon \mathcal{Z} \cap A$. Then $Z_A \cup Z'_A = (Z \cap A) \cup (Z' \cap A) = (Z \cup Z') \cap A$ where $Z_1 Z' \epsilon \mathcal{Z}$. Since $\mathcal{Z}$ is closed under finite unions, it follows that $Z_A \cup Z'_A \epsilon \mathcal{Z} \cap A$. Let $\{ Z^n_A \epsilon \mathcal{Z} \cap A : n\epsilon \mathbb{N} \}$ be a countable collection of subsets of $\mathcal{Z} \cap A$, Then

$\cap \{ Z^n_A : n\epsilon \mathbb{N} \} = \cap \{ Z^n \cap A : n\epsilon \mathbb{N} \} = \cap \{ Z^n : n\epsilon \mathbb{N} \} \cap A$ where each $Z^n \epsilon \mathcal{Z}$. Since $\mathcal{Z}$ is closed under countable intersections, it follows that $\cap \{ Z^n_A \epsilon \mathcal{Z}_A : n\epsilon \mathbb{N} \} \epsilon \mathcal{Z} \cap A$. Hence $\mathcal{Z} \cap A$ is a delta ring of sets.

(3)   Let $Z_A \epsilon \mathcal{Z} \cap A$ where $Z_A = Z \cap A$ with $Z\epsilon \mathcal{Z}$. Since $\mathcal{Z}$ is complement generated, $Z_A = Z \cap A = \cap \{ X \backslash Z^n : n\epsilon \mathbb{N} \} \cap A = \cap \{ A \backslash (Z^n \cap A) : n\epsilon \mathbb{N} \} = \cap \{ A \backslash Z^n_A : n\epsilon \mathbb{N} \}$ where $Z^n_A = Z^n \cap A \epsilon \mathcal{Z} \cap A$. Hence $\mathcal{Z} \cap A$ is complement generated.

The next lemma obtains similar results for properties defined on a topological space.

3.27 <u>Lemma</u>. <u>Let</u> X <u>be</u> <u>a</u> <u>topological</u> <u>space</u> <u>and</u> <u>let</u> A$\subset$X <u>such that</u> $\mathcal{Z} \cap A \neq \phi$. <u>The</u> <u>following</u> <u>statements</u> <u>are</u> <u>true</u>.

(1)   <u>If</u> $\mathcal{Z}$ <u>is</u> <u>a</u> <u>base</u> <u>for</u> <u>the</u> <u>closed</u> <u>sets</u> <u>in</u> X <u>then</u> $\mathcal{Z} \cap A$ <u>is</u> <u>a</u> <u>base</u> <u>for</u> <u>the</u> <u>closed</u> <u>sets</u> <u>in</u> A.

(2)   <u>If</u> $\mathcal{Z}$ <u>is</u> <u>a</u> <u>disjunctive</u> <u>collection</u> <u>in</u> X <u>then</u> $\mathcal{Z} \cap A$ <u>is a</u> <u>disjunctive</u> <u>collection</u> <u>in</u> A.

(3)   <u>If</u> $\mathcal{Z}$ <u>is</u> <u>a</u> <u>local</u> <u>base</u> <u>in</u> X <u>then</u> $\mathcal{Z} \cap A$ <u>is a</u> <u>local</u> <u>base</u> <u>in</u> A.

<u>Proof</u>: (1) Let F_A be a closed subset of A. Then $F_A = F \cap A$ where F is a closed subset of X. Since $\mathcal{Z}$ is a base for the closed sets, $F_A = F \cap A = \cap\{ Z^n \varepsilon \mathcal{Z} : n \varepsilon \mathbb{N}\} \cap A = \cap \{ Z^n \cap A : n \varepsilon \mathbb{N}\} = \cap\{ Z^n_A : n \varepsilon \mathbb{N}\}$ where each $Z^n_A \varepsilon \mathcal{Z} \cap A$. Hence $\mathcal{Z} \cap A$ is a base for the closed sets.

(2) Let F_A be a closed subset of A and let $x \varepsilon A \smallsetminus F_A$. Then $x \varepsilon F$ where F is a closed subset of X and $F_A = F \cap A$. Because $\mathcal{Z}$ is disjunctive, there exists $Z \varepsilon \mathcal{Z}$ such that $x \varepsilon Z$ and $Z \cap F = \emptyset$. It follows that $x \varepsilon Z_A = Z \cap A$ and $Z_A \cap F_A = \emptyset$. Hence $\mathcal{Z} \cap A$ is disjunctive.

(3) Let $x \varepsilon A$ and let $N_A(x)$ be a neighborhood of x in A. Then there is a neighborhood $N(x)$ of x in X such that $N(x) \cap A = N_A(x)$. Since $\mathcal{Z}$ is a local base, there is a $Z \varepsilon \mathcal{Z}$ such that $x \varepsilon$ int $Z \subset Z \subset N(x)$. It follows that $x \varepsilon (\text{int } Z) \cap A \subset Z \cap A \subset N(x) \cap A$ so that $x \varepsilon \text{int}_A Z_A \subset Z_A \subset N_A(x)$ where $Z \cap A = Z_A \varepsilon \mathcal{Z} \cap A$. Hence $\mathcal{Z} \cap A$ is a local base.

So far the properties that were imposed earlier on $\mathcal{Z}$ have traced down to a subset or subspace very easily. Unfortunately this is not the case for a normal collection. The next example which is similar to one in [28] demonstrates this fact.

Let $X = \mathbb{N} \cup \{ 0 \}$ and let $\mathcal{Z} = \{ Z \varepsilon \mathbb{P}(X) :$ Z is a singleton point, $X \smallsetminus Z$ is a singleton point, Z consists of all points of the form $4n+3$ for $n \varepsilon X$ and 0, or Z consists of all points of the form $4n$ for $n \varepsilon X \}$. It is readily verified that $\mathcal{Z}$ is a normal collection.

Let $A = \mathbb{N}$ and consider $\mathfrak{Z} \cap A$. The two sets $Z_1 = \{ 4n : n \in \mathbb{N} \}$ and $Z_2 = \{ 4n + 3 : n \in \mathbb{N} \}$ belong to $\mathfrak{Z} \cap A$ and $Z_1 \cap Z_2 = \phi$. However, Z_1 and Z_2 are not contained in complements of sets from $\mathfrak{Z} \cap A$ whose intersection is empty. Thus $\mathfrak{Z} \cap A$ is not normal.

The concept of nest generated is useful in obtaining meaningful conditions where $\mathfrak{Z} \cap A$ is normal. First, it must be shown that the property of nest generated traces to $\mathfrak{Z} \cap A$.

3.28 <u>Lemma</u>. <u>Let</u> X <u>be a non-empty set and let</u> $A \subset X$ <u>with</u> $\mathfrak{Z} \cap A \neq \phi$. <u>If</u> $\mathfrak{Z}$ <u>is nest generated then</u> $\mathfrak{Z} \cap A$ <u>is nest generated</u>.

<u>Proof</u>: Let $Z \cap A \in \mathfrak{Z} \cap A$. Since $\mathfrak{Z}$ is nest generated, there exist sequences $\{ Z_n : n \in \mathbb{N} \}$ and $\{ C_n : n \in \mathbb{N} \}$ such that $Z = \cap \{ Z_n : n \in \mathbb{N} \}$ and $C_{n+1} \subset Z_{n+1} \subset C_n \subset Z_n$ for each $n \in \mathbb{N}$. Then $Z \cap A = \cap \{ Z_n : n \in \mathbb{N} \} \cap A = \cap \{ Z_n \cap A : n \in \mathbb{N} \}$ and $C_{n+1} \cap A \subset Z_{n+1} \cap A \subset C_n \cap A \subset Z_n \cap A$ for each $n \in \mathbb{N}$. But $C_n \cap A = D_n$ where D_n is the complement of a $\mathfrak{Z} \cap A$ set. Hence, $\mathfrak{Z} \cap A$ is nest generated.

3.29 <u>Corollary</u>. <u>Let</u> A <u>be a non-empty subset of</u> X <u>such that</u> $\mathfrak{Z} \cap A \neq \phi$. <u>If</u> $\mathfrak{Z}$ <u>is a normal, complement generated delta ring, then</u> $\mathfrak{Z} \cap A$ <u>is a normal collection</u>.

<u>Proof</u>: By (3.25), $\mathfrak{Z}$ is nest generated. By (3.26) and (3.28), $\mathfrak{Z} \cap A$ is a nest generated delta ring. Hence, by (3.25), $\mathfrak{Z} \cap A$ is normal.

$\mathfrak{Z}$-CONTINUITY

The theory of Bourbaki filters can be utilized in determining whether a given function is continuous. Recall that for a function f from a space X into a space X' and a Bourbaki filter $\mathfrak{T}$ on X, the collection $f(\mathfrak{T})=\{f(F) : F\epsilon\mathfrak{T}\}$ is a filter base on X'. However, if $\mathfrak{Z}$ and $\mathfrak{Z}'$ are sub-collections of X and X' respectively and $\mathfrak{T}$ is a $\mathfrak{Z}$-filter the collection $f(\mathfrak{T})$ is not necessarily a $\mathfrak{Z}'$-filter because $f(F)$ is not in general a member of $\mathfrak{Z}'$. This section is concerned with relating $\mathfrak{Z}$-filters and continuity.

4.1 <u>Lemma</u>. <u>Let</u> $\mathfrak{Z}$ <u>and</u> $\mathfrak{Z}'$ <u>be collections of subsets of the topological spaces</u> X <u>and</u> X' <u>respectively and let</u> $\mathfrak{T}$ <u>be a</u> $\mathfrak{Z}$-<u>filter on</u> X. <u>If</u> f <u>is a function from</u> X <u>into</u> X' <u>satisfying</u> $f(F)\epsilon\mathfrak{Z}'$ <u>for all</u> $F\epsilon\mathfrak{T}$, <u>then</u> $f(\mathfrak{T})$ <u>is a</u> $\mathfrak{Z}'$-<u>filter</u> <u>base</u>.

<u>Proof</u>: Since $\mathfrak{T}$ is non-empty, the same is true of $f(\mathfrak{T})$. Also, for each $F\epsilon\mathfrak{T}$, $f(F) \neq \phi$ because f is a function. If $f(F_1)$ and $f(F_2)$ belong to $f(\mathfrak{T})$, then $F_1 \cap F_2 \epsilon \mathfrak{T}$ and $f(F_1 \cap F_2) \subset f(F_1) \cap f(F_2)$. By (2.4) $f(\mathfrak{T})$ is a $\mathfrak{Z}$-filter base.

The above result suggests a possible approach to obtaining $\mathfrak{Z}'$-filters. However, the fact that for a continuous function the pre-image of open sets and closed sets are, respectively, open sets and closed sets while the same need not be true of the image motivates the following definition.

4.2 <u>Definition</u>. A function f from a space X into a space X$'$ is called $\mathfrak{Z}$-<u>continuous</u> in case $f^{-1}[Z']\epsilon\mathfrak{Z}$ for each $Z'\epsilon\mathfrak{Z}'$.

4.3 <u>Lemma</u>. <u>If</u> $\mathfrak{Z}$ <u>and</u> $\mathfrak{Z}'$ <u>are</u> <u>bases</u> <u>for</u> <u>the</u> <u>closed</u> <u>sets</u> <u>of</u> X <u>and</u> X$'$ <u>and</u> <u>if</u> <u>the</u> <u>function</u> f : X → X$'$ <u>is</u> $\mathfrak{Z}$-<u>continuous</u>, <u>then</u> f <u>is</u> <u>continuous</u>.

<u>Proof</u>: Let F$'$ be a closed set in X$'$. Since $\mathfrak{Z}'$ is a base for the closed sets, $F' = \cap\{ Z'_\lambda\epsilon\mathfrak{Z}' : \lambda\epsilon\Lambda \}$. Then $f^{-1}[F'] = \cap\{ f^{-1}[Z'_\lambda] : \lambda\epsilon\Lambda \}$ which is closed because $f^{-1}[Z'_\lambda]\epsilon\mathfrak{Z}$ for each $\lambda\epsilon\Lambda$. Hence, f is continuous.

4.4 <u>Lemma</u>. <u>If</u> $\mathfrak{Z}$ <u>is</u> <u>the</u> <u>collection</u> <u>of</u> <u>all</u> <u>closed</u> <u>subsets</u> <u>of</u> X, <u>if</u> $\mathfrak{Z}'$<u>is</u> <u>a</u> <u>base</u> <u>for</u> <u>the</u> <u>closed</u> <u>sets</u> <u>of</u> X$'$, <u>and</u> <u>if</u> <u>the</u> <u>function</u> f : X→X$'$ <u>is</u> <u>continuous</u>, <u>then</u> f <u>is</u> $\mathfrak{Z}$-<u>continuous</u>.

<u>Proof</u>: For any closed set F$'$ in X$'$ it follows that $f^{-1}[F'] = \cap\{ f^{-1}[Z'_\lambda] : \lambda\epsilon\Lambda \}$. Since f is continuous, $f^{-1}[Z'_\lambda]$ is closed for each $\lambda\epsilon\Lambda$. Hence $f^{-1}[F']\epsilon\mathfrak{Z}$ and f is $\mathfrak{Z}$-continuous.

As the collection f($\mathcal{C}$) is used in Bourbaki filters for a filter base in the range space, the following collection plays a similar role for $\mathfrak{Z}$-filters. For a function f from X into X$'$ and a $\mathfrak{Z}$-filter $\mathcal{C}$ <u>denote</u> <u>the</u> <u>collection</u> { $Z'\epsilon\mathfrak{Z}' : f^{-1}[Z']\epsilon\mathcal{C}$} <u>by</u> $f^{\#}\mathcal{C}$.

4.5 <u>Lemma</u>. <u>Let</u> <u>the</u> <u>function</u> f : X→X$'$ <u>be</u> $\mathfrak{Z}$-<u>continuous</u> <u>and</u> <u>let</u> $\mathcal{C}$ <u>be</u> <u>a</u> $\mathfrak{Z}$-<u>filter</u>. <u>If</u> $f^{\#}\mathcal{C}$ <u>is</u> <u>non-empty</u>, <u>then</u> $f^{\#}\mathcal{C}$ <u>is</u> <u>a</u> $\mathfrak{Z}'$-<u>filter</u>.

Proof: The empty set does not belong to $f^{\#}\mathfrak{T}$ because $f(F) \neq \emptyset$ for any $F \varepsilon \mathfrak{T}$. Let Z_1' and Z_2' belong to $f^{\#}\mathfrak{T}$. Then $f^{-1}[Z_1']$ and $f^{-1}[Z_2']$ are members of $\mathfrak{T}$. Now, $f^{-1}[Z_1' \cap Z_2'] = f^{-1}[Z_1'] \cap f^{-1}[Z_2'] \varepsilon \mathfrak{T}$ and so $Z_1' \cap Z_2' \varepsilon f^{\#}\mathfrak{T}$. Let $Z_1' \varepsilon \mathfrak{Z}'$ contain a member Z' of $f^{\#}\mathfrak{T}$. Then $f^{-1}[Z_1'] \supset f^{-1}[Z']$ and $f^{-1}[Z'] \varepsilon \mathfrak{T}$. It follows that $Z_1' \varepsilon f^{\#}\mathfrak{T}$. Therefore, $f^{\#}\mathfrak{T}$ is a $\mathfrak{Z}'$-filter.

4.6 <u>Definition</u>. Let f be a function from X into X' and let $\mathfrak{T}$ be a $\mathfrak{Z}$-filter. <u>The point $y \varepsilon X'$ is called a limit point of f with respect to $\mathfrak{T}$ in case $f^{\#}\mathfrak{T}$ is a $\mathfrak{Z}'$-filter with limit point y.</u>

4.7 <u>Lemma</u>. <u>Let f be a function from X into X' and let $\mathfrak{T}$ be a $\mathfrak{Z}$-filter. If y is a limit point of f with respect to $\mathfrak{T}$, then for every neighborhood $N(y)$ of y there exists $F \varepsilon \mathfrak{T}$ such that $f(F) \subset N(y)$.</u>

Proof: Let $N(y)$ be a neighborhood of y. Since y is a limit point of $f^{\#}\mathfrak{T}$, there is a $Z' \varepsilon f^{\#}\mathfrak{T}$ such that $Z' \subset N(y)$ and $f^{-1}[Z'] = F \varepsilon \mathfrak{T}$. It follows that $f[F] \subset Z' \subset N(y)$.

The following corollary follows immediately from (4.7).

4.8 <u>Corollary</u>. <u>Let f be a function from X into X' and let $\mathfrak{Z}$ be a local base. If $f^{\#}\mathfrak{V}(x)$ is a $\mathfrak{Z}'$-filter with limit point $f(x)$, then f is continuous at x.</u>

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4.9 _Lemma._ _Let f be a function from X into X′ and let_
$\mathfrak{B}$ _and_ $\mathfrak{B}′$ _be local bases. If f is_ $\mathfrak{B}$-_continuous and_
continuous at x∈X, _then_ $f^{\#}\mathcal{V}(x)$ _is a_ $\mathfrak{B}′$-_filter with limit_
point f(x).

Proof: Let N(f(x)) be a neighborhood of f(x). Then
there is a Z′∈$\mathfrak{B}′$ such that f(x)∈int Z′ ⊂ Z′⊂N(f(x)). Since
f is continuous, it follows that f^{-1}[int Z′] is a neighbor-
hood of x. Then there is a Z∈$\mathfrak{B}$ such that
x∈ int Z ⊂ Z ⊂f^{-1}[int Z′] and so Z∈$\mathcal{V}(x)$. Since f is
$\mathfrak{B}$ ‑continuous, f^{-1}[Z′]∈ $\mathcal{V}(x)$. Hence Z′∈$f^{\#}\mathcal{V}(x)$ so by (4.5),
$f^{\#}\mathcal{V}(x)$ is a $\mathfrak{B}′$-filter. It follows that f(x) is a limit
point of $f^{\#}\mathcal{V}(x)$ because Z′⊂N(f(x)).

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13. ABSTRACT

This manuscript is a collection of many of the known results in the theory of generalized filters ($\mathfrak{R}$ -filters) as well as an extension of some of the work in this area. The relation of $\mathfrak{R}$ -filters to compactifications and realcompactifications is given special attention. Special emphasis is also given to the concept of tracing.

After the necessary preliminaries are disposed of, Section I motivates the study of $\mathfrak{R}$ -filters by discussing the collection of zero-sets and then the Wallman compactification. The concept of realcompactification is also mentioned. Section II introduces the concept of $\mathfrak{R}$ -filters and exhibits many of the elementary facts about these generalized filters.

The beginning of Section III deals with the convergence of $\mathfrak{R}$ -filters. The Frink or Wallman-type compactification of a Tychonoff space is presented in detail. The generalization of this method in constructing realcompactifications in Tychonoff spaces is also discussed. The Section closes with the presentation of some tracing results. Finally, Section IV investigates the concept of $\mathfrak{R}$ -continuity.

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